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# The Numerical Solution of Fractional Differential-Algebraic Equations (FDAEs) by Haar Wavelet Method

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**Abstract:** In this paper, the numerical solution of Fractional Differential-Algebraic Equations (FDAEs) is considered by Haar wavelet method. We derive the Haar wavelet operational matrix of the fractional order integration and by using it to solve the Fractional Differential-Algebraic Equations. The results obtained are in good-agreement with the exact solutions. It is shown that the technique used here is effective and easy to apply.
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## 1. Introduction

Fractional modeling in differential equations has gained considerable popularity and importance during the past three decades or more. Besides, Differential-Algebraic Equations (DAEs) have been successfully used to characterize for many physical and engineering topics such as polymer physics, fluid flow, electromagnetic theory, dynamics of earthquakes, rheology, viscoelastic materials, viscous damping and seismic analysis. Also differential-algebraic equations with fractional order have been made in some mathematical models in recent times. As known, fractional differential-algebraic equations usually do not have exact solutions. Therefore, approximations and numerical techniques must be used for them and also the solution of these equations has been a subject for many researchers.[1-9]

In this paper, we want to show the Haar wavelet method to solve the fractional order differential-algebraic equations. Firstly, we derive Haar wavelet operational matrix of the fractional order integration and then we use the Haar wavelet operational matrices of the fractional order integration to completely transform the fractional order systems into algebraic systems of equations. Finally, we solve this transformed complicated algebraic equations system by the software.

A fractional differential-algebraic equation (FDAE) with the initial conditions is defined as the form below [10]

$$\boldsymbol{D_*^{\alpha_i} x_i(t) = f_i(t, x_1, x_2, \dots x_n, x_1', x_2', \dots x_n')}$$
$$i = 1,2,3, \dots n-1, \quad t \geq 0\ , 0 < \alpha_i \leq 1$$
$$\boldsymbol{g(t, x_1, x_2, \dots x_n) = 0}$$
$$x_i(0) = \alpha_i \qquad i = 1,2,3, \dots, n \quad (1)$$

## 2. Basic definitions

There are several definitions of a fractional derivative of order $\alpha > 0$ [11], for example. Riemann-Liouville, Caputo, Grünwald-Letnikov, and the generalized functions approach. The most common definitions are Riemann-Liouville and Caputo. We give some basic definitions and properties of fractional calculus theory which are used in this paper.

**Definition 2.1.** A real function $f(x), x > 0.$ is said to be in the space $C_\mu, \mu \in R$ if there exists a real number $p > \mu$ such that $f(x) = x^p f_I(x)$, where $f_I(x) \in C[0, \infty)$. Clearly, $C_\mu \subset C_\beta$ if $\beta < \mu$.

**Definition 2.2.** A function $f(x), x < 0.$ is said to be in the space $C_\mu^m, m \in N \cup \{0\}$ if $f^{(m)} \in C_\mu$

**Definition 2.3.** The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ of a function, $f \in C_\mu,\ \mu \geq -1$, is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t) dt\ , \alpha > 0, x > 0 \quad (2)$$
$$J^0 f(x) = f(x). \quad (3)$$

The properties of the operator $f^\alpha$ can be found in [12, 13]. We make use of the followings.

For $f \in C_\mu,\ \ \mu \geq -1,\ \ \alpha, \beta \geq 0$ and $\gamma > -1$:

1. $J^\alpha J^\beta f(x) = J^{\alpha+\beta} f(x)$ (4)
2. $J^\alpha J^\beta f(x) = J^\beta J^\alpha f(x)$ (5)
3. $J^\alpha x^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}$ (6)

The Riemann- Liouville fractional derivative has some disadvantages making a model for real-world subjects using fractional differential and fractional differential-algebraic equations. Therefore, we sometimes use a modified fractional differential operator $D_*^\alpha$ introduced by Caputo's work on the theory of viscoelasticity [14].

**Definition 2.4.** The fractional derivative of $f(x)$ by Caputo is defined as

$$D_*^\alpha f(x) = J^{m-\alpha} D^m f(x)$$
$$= \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha-1} f^{(m)}(t)dt, \quad (7)$$

for $m-1 < \alpha \leq m,\ m \in N,\ x > 0,\ f \in C_{-1}^m$.
Also, we need here two basic properties.

**Lemma 2.1.** If $m-1 < \alpha \leq m,\ \ m \in N$ and $f \in C_\mu^m, m \geq -1$, then

1. $D_*^\alpha J^\alpha f(x) = f(x)$ (8)
2. $J^\alpha D_*^\alpha f(x) =$
$f(x) - \sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ \ x > 0.$ (9)

## 3. Haar wavelet operational matrix of fractional order integration

### 3.1 Haar Function

The orthogonal basis $\{h_n(t)\}$ of Haar wavelets for the Hilbert space $L_2[0,1]$ consists of

$$h_n(t) = h_1(2^j t - k),$$
$$n = 2^j + k,\ j \geq 0,\ 0 \leq k \leq 2^j \quad n, j, k \in Z \quad (10)$$

where

$$h_0(t) = 1, \quad 0 \leq t < 1,$$

$$h_1(t) = \begin{cases} 1, & 0 \leq t < 0.5, \\ -1, & 0.5 \leq t < 1. \end{cases} \quad (11)$$

each Haar wavelet $h_n$ has the support
$$(2^{-j}k,\ 2^{-j}(k+1)),$$
so that it is zero elsewhere in the interval $[0, 1)$. As might be expected, as $n$ increases, the Haar wavelets become more and more localized. That is, $\{h_n(t)\}$ are like a local basis.

Any function $f(t) \in L_2([0,1])$ can be expanded in Haar series

$$f(t) = \sum_{i=0}^{\infty} c_i h_i(t)$$
$$n = 2^j + k,\ j \geq 0,\ 0 \leq k \leq 2^j \quad (12)$$

where the Haar coefficients $c_i$, $i = 1,2, ...$, are written by

$$c_i = 2^j \int_0^1 f(t) h_i(t) dt \quad (13)$$

which are determined such that the following integral square error $\varepsilon$ is minimized

$$\varepsilon = \int_0^1 \left[ f(t) - \sum_{i=0}^{m-1} c_i h_i(t) \right]^2 dt,$$
$$m = 2^j$$
$$j \in \{0\} \cup N \quad (14)$$

By using the orthogonal property of Haar wavelet

$$\int_0^1 h_l(t) h_i(t) dt = \begin{cases} 2^{-j}, & i = l, \\ 0, & i \neq l. \end{cases}$$

The series in Eq. (13) has infinite number of terms. If $f(t)$ is piecewise constant or may be approximated as piecewise constant, then the sum in Eq. (13) may be terminated after $m$ terms, that is [15]

$$f(t) \approx \sum_{i=0}^{m-1} c_i h_i(t)\, m = C_m^T H_m(t) = \hat{f}(t) \quad (15)$$

where $m = 2^j$, the superscript $T$ indicates transposition, $\hat{f}(t)$ denotes the truncated sum. The Haar coefficient vector $C_m$ and Haar function vector $H_m(t)$ are defined as

$$C_m \triangleq [c_0, c_1, \dots, c_{m-1}]^T \quad (16)$$

$$H_m(t) \triangleq [h_0(t), h_1(t), \dots, h_{m-1}(t)]^T \quad (17)$$

Selecting the collocation points as following

$$t_i = \frac{(2i-1)}{2m}, i = 1,2, \dots, m \quad (18)$$

We defined the $m -$ square Haar matrix $\Phi_{m\times m}$ as:

$$\Phi_{m\times m} \triangleq \left[ H_m\left(\frac{1}{2m}\right) \ \ H_m\left(\frac{3}{2m}\right) \ \cdots \ H_m\left(\frac{2m-1}{2m}\right) \right] \quad (19)$$

For example, when $m = 8$, the Haar matrix is expressed as

$$\Phi_{8\times8}=\begin{bmatrix}1&1&1&1&1&1&1&1\\1&1&1&-1&-1&-1&-1&-1\\1&1&-1&0&0&0&0&0\\0&0&0&1&1&1&-1&-1\\1&-1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&1&-1&0&0\\0&0&0&0&0&0&-1&-1\end{bmatrix} \tag{20}$$

Correspondingly, we have

$$\hat{f}_m=\left[\hat{f}\left(\frac{1}{2m}\right)\ \hat{f}\left(\frac{3}{2m}\right)\cdots\hat{f}\left(\frac{2m-1}{2m}\right)\right]=C_m^T\Phi_{\mathrm{m\times m}} \tag{21}$$

Because the m-square Haar matrix $\Phi_{\mathrm{m\times m}}$ is an invertible matrix, the Haar coefficient vector $C_m^T$ can be gotten by [15]

$$C_m^T=\hat{f}_m\Phi^{-1}{}_{\mathrm{m\times m}} \tag{22}$$

### 3.2. Operational matrix of the fractional order integration

The integration of the $H_m(t)$ defined in Eq. (17) can be approximated by Haar series with Haar coefficient matrix P [16].

$$\int_0^t H_m(\tau)d\tau\approx \mathrm{P}_{\mathrm{m\times m}}\,H_m(t) \tag{23}$$

where the m-square matrix P is called the Haar wavelet operational matrix of integration [16]. Our purpose is to derive the Haar wavelet operational matrix of the fractional order integration. For this purpose, we use the definition of Riemann–Liouville fractional order integration, as below [13]

$$(I^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(1-t)^{\alpha-1}f(\tau)d\tau=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t) \tag{24}$$

where $\alpha\in R$ is the order of the integration, $\Gamma(\alpha)$ is the Gamma function and $t^{\alpha-1}*f(t)$ denotes the convolution product of $t^{\alpha-1}$ and $f(t)$. Now if $f(t)$ is expanded in Haar functions, as shown in Eq. (15), the Riemann–Liouville fractional integration becomes

$$(I^\alpha f)(t)=\frac{1}{\Gamma(\alpha)}t^{\alpha-1}*f(t)\approx C_m^T\frac{1}{\Gamma(\alpha)}\{t^{\alpha-1}*H_m(t)\} \tag{25}$$

Thus if $t^{\alpha-1}*f(t)$ can be integrated, then expanded in Haar functions, the Riemann Liouville fractional order integration is solved via the Haar functions.

Also, we define a $m$ –set of Block Pulse Functions (BPF) as:

$$b_i(t)=\begin{cases}1, 1/m\le t<(1+m)/m,\\ \quad 0,\ \text{otherwise}\end{cases} \tag{26}$$

where $i=0,1,2,\cdots,(m-1)$,

The functions $b_i(t)$ are disjoint and orthogonal. That is,

$$b_i(t)b_l(t)=\begin{cases}0, & i\neq l\\ b_i(t), & i=l\end{cases} \tag{27}$$

$$\int_0^1 b_i(\tau)b_l(\tau)d\tau=\begin{cases}0, & i\neq l\\ 1/m, & i=l\end{cases} \tag{28}$$

As seen the Haar functions are piecewise constant, and so it can be expanded into an m-term block pulse functions (BPF) as

$$H_m(t)=\Phi_{m\times m}B_m(t) \tag{29}$$

where $B_m(t)\triangleq [b_0(t)\ \ b_0(t)\ \cdots\ b_i(t)\cdots b_{m-1}(t)]^T$(30)

In Ref. [17], Kilicman and Al Zhour have given the Block Pulse operational matrix of the fractional order integration $F^\alpha$ as following

$$(I^\alpha B_m)(t)\approx F^\alpha B_m(t) \tag{31}$$

where

$$F^\alpha=\frac{1}{m^\alpha}\frac{1}{\Gamma(\alpha+2)}\begin{bmatrix}1&\xi_1&\xi_2&\cdots&\xi_{m-1}\\0&1&\xi_1&\cdots&\xi_{m-2}\\0&0&1&\cdots&\xi_{m-3}\\ & & & & \\0&0&0&\ddots&\vdots\\0&0&0&0&1\end{bmatrix} \tag{32}$$

With $\xi_k=(k+1)^{\alpha+1}-2k^{\alpha+1}+(k-1)^{\alpha+1}$ (33)

Next, we derive the Haar wavelet operational matrix of the fractional order integration.
Let

$$(I^\alpha H_m)(t)\approx P^\alpha_{m\times m}\mathrm{H_m(t)} \tag{34}$$

where the $m-$ square matrix $P^{\alpha}_{m\times m}$ is called the Haar wavelet operational matrix of the fractional order integration. Using Eqs. (29)(30) and (31), we get

$$(I^{\alpha}H_m)(t) \approx (I^{\alpha}\Phi_{\mathrm{m\times m}}B_m)(t) = \Phi_{\mathrm{m\times m}}(I^{\alpha}B_m)(t) \approx \Phi_{\mathrm{m\times m}}F^{\alpha}B_m(t) \quad (35)$$

From Eqs. (34) and (35) we get

$$P^{\alpha}_{m\times m}H_m(t) = P^{\alpha}_{m\times m}\Phi_{\mathrm{m\times m}}B_m(t) = \Phi_{\mathrm{m\times m}}F^{\alpha}B_m(t) \quad (36)$$

Then, the Haar wavelet operational matrix of the fractional order integration $P^{\alpha}_{m\times m}$ is given by

$$P^{\alpha}_{m\times m} = \Phi_{\mathrm{m\times m}}F^{\alpha}\Phi^{-1}_{\mathrm{m\times m}} \quad (37)$$

For example, let $\alpha = 0.5$, $m = 8$, the operational matrix $P^{\alpha}_{m\times m}$ is computed below [15]

$$P^{0.5}_{8\times 8} = \begin{bmatrix} 0.7523 & -0.2203 & -0.1558 & -0.0820 & -0.1102 & -0.0580 & -0.0447 & -0.0377 \\ 0.2203 & 0.3116 & -0.1558 & 0.2296 & -0.1102 & -0.0580 & 0.1756 & 0.0782 \\ 0.0410 & 0.1148 & 0.2203 & -0.0350 & -0.1102 & 0.1623 & -0.0389 & -0.0063 \\ 0.0779 & -0.0779 & 0 & 0.2203 & 0 & 0 & -0.1102 & 0.1623 \\ 0.0094 & 0.0196 & 0.0812 & -0.0032 & 0.1558 & -0.0247 & -0.0026 & -0.0009 \\ 0.0112 & 0.0439 & -0.0551 & -0.0194 & 0 & 0.1558 & -0.0247 & -0.0026 \\ 0.0145 & -0.0145 & 0 & 0.0812 & 0 & 0 & 0.1558 & -0.0247 \\ 0.0275 & -0.0275 & 0 & -0.0551 & 0 & 0 & 0 & 0.1558 \end{bmatrix}$$

**4. Numerical Applications**

Showing the efficiency of the method, we consider the following fractional differential-algebraic equations. All the numerical results were obtained by using the software MAPLE.

**Example 4.1.** We consider the following fractional -algebraic equation.

$$\begin{gathered} D^{\alpha}x(t) - tDy(t) + x(t) - (1+t)y(t) = 0 \\ 0 < \alpha \le 1 \\ y(t) - sint = 0 \end{gathered} \quad (38)$$

with initial conditions $x(0) = 1,\ y(0) = 0$ and exact solutions $x(t) = e^{-t} + tsint,\ y(t) = sint$ when $\alpha = 1$

Firstly, we add $Dx(t)$ to both sides of the first equality for applicability.

$$\begin{gathered} Dx(t) + D^{\alpha}x(t) - tDy(t) + x(t) - (1+t)y(t) \\ = -e^{-t} + sint + tcost \\ y(t) - sint = 0 \end{gathered} \quad (39)$$

Now, let

$$Dx(t) = R^T H_m(t) \quad \text{and} \quad Dy(t) = K^T H_m(t) \quad (40)$$

together with the initial states, then we have:

$$D^{\alpha}x(t) = R^T P^{1-\alpha}_{m\times m}H_m(t) \quad (41)$$

$$x(t) = R^T P^{1}_{m\times m}H_m(t) + \underbrace{1}_{x(0)} \quad (42)$$

$$y(t) = K^T P^{1}_{m\times m}H_m(t) + \underbrace{0}_{y(0)} \quad (43)$$

Similarly, $f(t) = -e^{-t} + sint + tcost$ may be expanded by the Haar functions as follows

$$f(t) = f^T_m H_m(t), \quad g(t) = g^T_m H_m(t) \quad (44)$$

Substituting Eqs. (40-41-42-43-44) into (39), we get

$$\begin{aligned} R^T H_m(t) + R^T P^{1-\alpha}_{m\times m}H_m(t) - tK^T H_m(t) \\ + R^T P^{1}_{m\times m}H_m(t) \\ + -(1-t)K^T P^{1}_{m\times m}H_m(t) \\ = f^T_m H_m(t) \end{aligned}$$

$$K^T P^{1}_{m\times m}H_m(t) = g^T_m H_m(t) \quad (45)$$

Hereby, Eq. (38) has been transformed into a system of algebraic equations. Substituting values of solving the system of algebraic equations, we can obtain the coefficients $R^T_m$ . Then using Eq. (42), we can get $x(t)$. The numerical result for $m = 16$ is shown in Table 1 and Fig 1. The numerical solution is in perfect agreement with the exact solutions.

**Example 4.2.** We consider the following fractional differential-algebraic equation.

$$\begin{gathered} D^{\alpha}x(t) + x(t) - y(t) = -sint \\ x(t) + y(t) = e^{-t} + sint \\ 0 < \alpha \le 1 \end{gathered} \quad (46)$$

with initial conditions $x(0) = 1,\ y(0) = 0$ and exact solutions $x(t) = e^{-t},\ y(t) = sint$ when $\alpha = 1$

Firstly, we add $Dx(t)$ to both sides of the second equality for applicability as above.

$$Dx(t) + D^{\alpha}x(t) + x(t) - y(t) = -sint - e^{-t}$$

$$x(t) + y(t) = e^{-t} + sint \tag{47}$$

Now, let

$$Dx(t) = U^T H_m(t) \quad \text{and} \quad Dy(t) = V^T H_m(t) \tag{48}$$

together with the initial states, then we have:

$$D^{\alpha}x(t) = U^T P^{1-\alpha}_{m\times m} H_m(t) \tag{49}$$

$$x(t) = U^T P^{1}_{m\times m} H_m(t) + \underbrace{1}_{x(0)} \tag{50}$$

$$y(t) = V^T P^{1}_{m\times m} H_m(t) + \underbrace{0}_{y(0)} \tag{51}$$

Similarly,

$$j(t) = e^{-t} + sint$$

and

$$n(t) = -e^{-t} - sint$$

may be expanded by the Haar functions as follows

$$j(t) = j^T_m H_m(t), \; n(t) = n^T_m H_m(t) \tag{52}$$

Substituting Eqs. (48-49-50-51-52) into (47), we get

$$U^T H_m(t) + U^T P^{1-\alpha}_{m\times m} H_m(t) + U^T P^{1}_{m\times m} H_m(t) + 1 - V^T P^{1}_{m\times m} H_m(t) = n^T_m H_m(t)$$

$$U^T P^{1}_{m\times m} H_m(t) + 1 + V^T P^{1}_{m\times m} H_m(t) = j^T_m H_m(t) \tag{53}$$

Hence, Eq. (46) has been transformed into an algebraic equations system. Solving this system, we can find the coefficients $U^T_m$ . Then using Eq. (50), we can get $x(t)$. The numerical result for $m = 16$ is shown in Table 2 and Fig 2. The numerical solution is in perfect agreement with the exact solutions.

Table 1 and Fig 1 shows the approximate solutions for Ex. (4.1) obtained for different values of $\alpha$. The results are in good agreement with the results of the exact solutions.

**Table 1.** Numerical results of the solution in **Example 4.1**

| ***m=16*** | $\boldsymbol{\alpha = 0,5}$ | $\boldsymbol{\alpha = 0,7}$ ! | $\boldsymbol{\alpha = 1}$ | |
|---|---|---|---|---|
| $\boldsymbol{t}$ | $\boldsymbol{x(t)}$ | $\boldsymbol{x(t)}$ | $\boldsymbol{x(t)}$ | $\boldsymbol{x_{exact}(t)}$ |
| **0.0** | **1.00000000** | **1.00000000** | **1.00000000** | **1.00000000** |
| **0.1** | **0.76419238** | **0.84829862** | **0.91482076** | **0.91482076** |
| **0.2** | **0.75450963** | **0.81669721** | **0.85846462** | **0.85846462** |
| **0.3** | **0.78941616** | **0.79789987** | **0.82947428** | **0.82947428** |
| **0.4** | **0.85248504** | **0.82408711** | **0.82608738** | **0.82608739** |
| **0.5** | **0.93142468** | **0.87591437** | **0.84624343** | **0.84624343** |
| **0.6** | **1.02320417** | **0.94545816** | **0.88759712** | **0.88759712** |
| **0.7** | **1.12579058** | **1.02917465** | **0.94753769** | **0.94753768** |
| **0.8** | **1.22732910** | **1.03284936** | **1.02321384** | **1.02321384** |
| **0.9** | **1.33439358** | **1.22342656** | **1.11156389** | **1.11156388** |
| **1.0** | **1.43375428** | **1.32697591** | **1.20935045** | **1.20935043** |

**Table 2.** Numerical results of the solution in **Example 4.2**

| ***m=16*** | $\alpha = 0,5$ | $\alpha = 0,7$ ! | $\alpha = 1$ | |
|---|---|---|---|---|
| $t$ | $x(t)$ | $x(t)$ | $x(t)$ | $x_{exact}(t)$ |
| **0.0** | **1.00000000** | **1.00000000** | **1.00000000** | **1.00000000** |
| **0.1** | **0.76089102** | **0.83739314** | **0.90483742** | **0.904837418** |
| **0.2** | **0.69092618** | **0.74943905** | **0.81873076** | **0.818730753** |
| **0.3** | **0.63965018** | **0.68161287** | **0.74081822** | **0.740818220** |
| **0.4** | **0.59708775** | **0.62503222** | **0.67032005** | **0.670320046** |
| **0.5** | **0.55999258** | **0.57601215** | **0.60653066** | **0.606530659** |
| **0.6** | **0.52688937** | **0.53262381** | **0.54881164** | **0.548811636** |
| **0.7** | **0.49696399** | **0.49371279** | **0.49658531** | **0.496585303** |
| **0.8** | **0.46970219** | **0.45851974** | **0.44932896** | **0.449328964** |
| **0.9** | **0.44474478** | **0.42650758** | **0.40656966** | **0.406569659** |
| **1.0** | **0.42182073** | **0.39727361** | **0.36787944** | **0.367879441** |

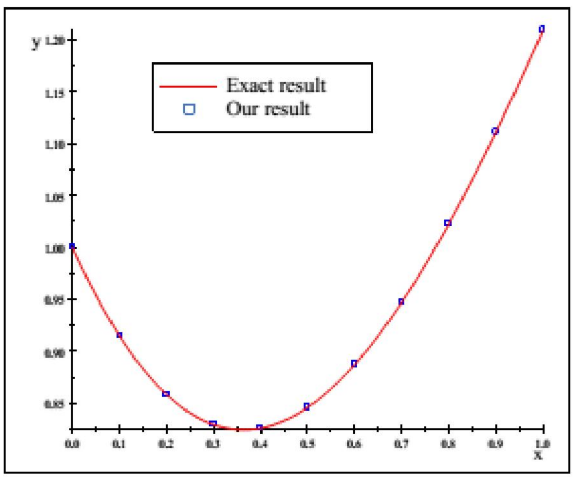


**Fig 1.** Graph of the numerical results for **Example 4.1**

Table 2 and Fig 2 shows the approximate solutions for Ex. (4.2) obtained for different values of $\alpha$. The results are in good agreement with the results of the exact solutions.

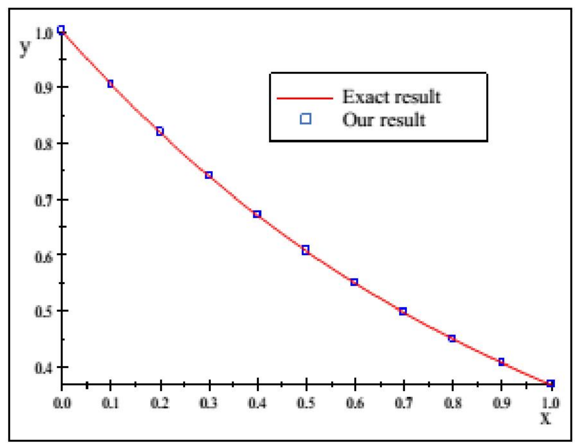


**Fig 2.** Graph of the numerical results for **Example 4.2**

**5. Conclusion**

In this paper, the Haar wavelet method has been extended to solve fractional differential-algebraic equations (FDAEs). The results of the method are in good agreement with obtained by exact solutions. The study show that the method is reliable techniques to handle fractional differential–algebraic equations, and the method offer significant advantages in terms of straightforward applicability, computational effectiveness, and accuracy

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