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Some Innovative Types of Fuzzy Ideals in *AG*-groupoids

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Abstract: AG-groupoids (non-associative structure) are basic structures in Flocks theory. This theory mainly focuses on distance optimization, motion replication, leadership maintenance with wide range of applications in physics and biology. In this paper, we define some new types of fuzzy ideals of AG-groupoids called (α,β)-fuzzy bi-ideals, (α,β)-fuzzy interior ideals, $(\overline{\beta},\overline{\alpha})$-fuzzy bi-ideals and $(\overline{\beta},\overline{\alpha})$-fuzzy interior ideals of AG-groupoids where $\alpha,\beta \in \{\in_\gamma, q_\delta, \in_\gamma \vee q_\delta, \in_\gamma \wedge q_\delta\}$ and $\overline{\alpha},\overline{\beta} \in \{\overline{\in}_\gamma, \overline{q}_\delta, \overline{\in}_\gamma \vee \overline{q}_\delta, \overline{\in}_\gamma \wedge \overline{q}_\delta\}$ with $\alpha \neq \in_\gamma \wedge q_\delta$ and $\overline{\beta} \neq \overline{\in}_\gamma \wedge \overline{q}_\delta$. An important milestone of this paper is to provide the connection between classical algebraic structures (ordinary bi-ideals, interior ideals) and new types of fuzzy algebraic structures $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy interior ideals). Special attention is given to $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideals.

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1. Introduction

The fuzzification of algebraic structures like semigroups, ordered semigroups, AG-groupoids, hemirings, near-rings, BCK/BCI-algebra in terms of several types of fuzzy ideals and investigation of new types of fuzzy ideals is a central focus for the researcher nowadays. Due to the diverse applications of such characterizations in many applied branches such as control engineering, computer science, Flocks theory, fuzzy coding theory, fuzzy finite state machines and fuzzy automata, the efforts made by mathematicians to investigate some new types of fuzzy ideals which can fill out the gap laying in the latest study of ideal theory. Mordeson et al. [**1**] presented an up to date account of fuzzy sub-semigroups and fuzzy ideals of a semigroup. Kuroki [**2**] introduced the notion of fuzzy bi-ideals in semigroups. Kehayopulu applied the fuzzy concept in ordered semigroups and studied some properties of fuzzy left (right) ideals and fuzzy filters in ordered semigroups (see [**3**]). Fuzzy implicative and Boolean filters of R_0-algebra were initiated by Liu and Li (see [**4**]).

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The concept of a quasi-coincidence of a fuzzy point with a fuzzy set, which is mentioned in [**5, 6**], played a significant role to generate some different types of fuzzy subgroups. The notion of (α,β)-fuzzy subgroups by using the *belongs to* relation $(\in)$ and *quasi-coincidence with* relation (q) between a fuzzy point and a fuzzy subgroup is introduced by Bhakat and Das [**5**]. In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroup [**7**]. This concept opened a new dimension for the researchers to investigate similar type of generalizations of the existing fuzzy subsystems of other algebraic structures. In algebra, the concept of (α,β)-fuzzy sets was introduced by Davvaz in [**8**], where $(\in, \in \vee q)$-fuzzy subnearrings (ideals) of a nearring were initiated and studied. With this objective in view, Ma et al. in [**9**], introduced the interval valued $(\in, \in \vee q)$-fuzzy ideals of pseudo-MV algebras and gave some important results of pseudo-MV algebras. Jun and Song [**10**] discussed general forms of fuzzy interior ideals in semigroups, also see [**11**]. Kazanci and Yamak introduced the concept of a generalized fuzzy bi-ideal in semigroups [**12**] and gave some properties of fuzzy bi-ideals in terms of $(\in,\in\vee q)$ -fuzzy bi-ideals. Jun et al. [**11**] gave the concept of a generalized fuzzy bi-ideal in ordered semigroups and characterized regular ordered semigroups in terms of this notion. Davvaz et al. used the idea of generalized fuzzy sets in hyperstructures and introduced different generalized fuzzy subsystems of hyperstructures, e.g., see references. In [**13**], Ma et al. introduced the concept of a generalized fuzzy filter of R_0-algebra and provided some properties in terms of this notion. Many other researchers used the idea of generalized fuzzy sets and gave several characterizations results in different branches of algebra see references. The concept of an (α,β)-fuzzy interior ideals in ordered semigroups was first introduced by Khan and Shabir in [**14**], where some basic properties of (α,β)-fuzzy interior ideals were discussed. Khan et al. [**15, 16**] characterized ordered semigroups in terms of some new types of fuzzy bi-ideals and fuzzy generalized bi-ideals.

Beside this, the concept of a *left almost semigroup* (LA-semigroup) [**17**], was first introduced by Kazim and Naseeruddin in 1972. In [**18**], the same structure was called a *left invertive groupoid.* Protic and Stevanivic [**19**] called it an *Abel-Grassmann's groupoid* abbreviated as AG-groupoid, a groupoid whose elements satisfy the *left invertive law*: $(ab)c=(cb)a$ for all $a,b,c\in G$. An AG-groupoid is the midway structure between a *commutative semigroup* and a *groupoid* [**20**]. It is a useful non-associative structure with wide range of applications in the theory of flocks [**21**]. In an AG-groupoid the *medial law*, $(ab)(cd)=(ac)(bd)$ for all $a,b,c,d\in G$ (see [**17**]) holds. If there exists an element e in an AG-groupoid G such that $ex=x$ for all $x\in G$ then G is called an *AG-groupoid with left identity e*. If an AG-groupoid G has the *right identity* then G is a *commutative monoid*. If an AG-groupoid G contains left identity then $(ab)(cd)=(dc)(ba)$ holds for all $a,b,c,d\in G$. Also $a(bc)=b(ac)$ holds for all $a,b,c\in G$.

Recently, Yin and Zhan [**22**] introduced more general forms of $(\in,\in\vee q)$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy (implicative, positive implicative and fantastic) filters of BL-algebras and defined $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy (implicative, positive implicative and fantastic) filters and $(\overline{\in_\gamma},\overline{\in_\gamma}\vee\overline{q_\delta})$-fuzzy (implicative, positive implicative and fantastic) filters of BL-algebras and gave some interesting results in terms of these notions.

In this paper, we generalize the concept of $(\in,\in\vee q)$-fuzzy bi-ideals given in [**23**], $(\in,\in\vee q)$-fuzzy left (right) ideals and $(\in,\in\vee q)$-fuzzy interior ideals given in [**24**] and define (α,β)-fuzzy bi-ideals, (α,β)-fuzzy interior ideals, $(\overline{\beta},\overline{\alpha})$-fuzzy bi-ideals and $(\overline{\beta},\overline{\alpha})$-fuzzy interior ideals in AG-groupoids where $\alpha,\beta\in\{\in_\gamma,q_\delta,\in_\gamma\vee q_\delta,\in_\gamma\wedge q_\delta\}$ and $\overline{\alpha},\overline{\beta}\in\{\overline{\in}_\gamma,\overline{q}_\delta,\overline{\in}_\gamma\vee\overline{q}_\delta,\overline{\in}_\gamma\wedge\overline{q}_\delta\}$ with $\alpha\neq\in_\gamma\wedge q_\delta$ and $\overline{\beta}\neq\overline{\in}_\gamma\wedge\overline{q}_\delta$. Special attention is given to $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideals, $(\overline{\in}_\gamma,\overline{\in}_\gamma\vee\overline{q}_\delta)$-fuzzy bi-ideals, $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideals and $(\overline{\in}_\gamma,\overline{\in}_\gamma\vee\overline{q}_\delta)$-fuzzy bi-ideals. Some interesting results in terms of these new types of notions are also obtained.

2. Preliminaries

The present section contains some fundamental concepts on AG-groupoids which are essential for this paper.

Through out the paper G denote an AG-groupoid unless otherwise stated.

For subsets A,B of an AG-groupoid G, we denote by $AB=\{ab\in G\mid a\in A,b\in B\}$. A non-empty subset A of G is called a *left (right) ideal* [**25**] of G if $GA\subseteq A(AG\subseteq A)$. A nonempty subset A of an AG-groupoid G is called an *AG-subgroupoid* of G if $A^2\subseteq A$. An AG-subgroupoid A of G is called a *bi-ideal* [**25**] of G if $(AG)A\subseteq A$. An AG-subgroupoid A of G is called an *interior ideal* [**25**] of G if $(GA)G\subseteq A$.

Now, we give some fuzzy logic concepts.

Let G be an AG-groupoid. By a fuzzy subset μ of an AG-groupoid G, we mean a mapping, $\mu: G\to[0,1]$. We denote by $\mu(G)$ the set of all fuzzy subsets of G. The order relation "$\subseteq$" on $\mu(G)$ is defined as follows:

$\mu_1\subseteq\mu_2$ if and only if $\mu_1(x)\leq\mu_2(x)$
for all $x\in G$ and for all $\mu_1,\mu_2\in\mu(G)$.

A fuzzy subset μ of G is called a f*uzzy AG-subgroupoid* [**26**] if $\mu(xy)\geq\mu(x)\wedge\mu(y)$ for all $x,y\in G$. μ is called a fuzzy *left (right) ideal* of G if $\mu(xy)\geq\mu(y)(\mu(xy)\geq\mu(x))$ for all $x,y\in G$. μ is called a *fuzzy ideal* of G if it is both a fuzzy left and right ideal of G.

Let G be an AG-groupoid and μ a fuzzy subset of G. Then μ is called a *fuzzy bi-ideal* [**26**] of G, if it satisfies the following conditions:

(1) $(\forall x,y\in G)(\mu(xy)\geq\min\{\mu(x),\mu(y)\})$.

(2) $(\forall x,y,z\in G)(\mu((xy)z)\geq\min\{\mu(x),\mu(z)\})$.

A fuzzy subset μ of G is called a *fuzzy interior ideal* [**26**] of G if it satisfies the following conditions:

(1) $(\forall x,y\in G)(\mu(xy)\geq\min\{\mu(x),\mu(y)\})$.

(2) $(\forall x,a,y\in G)(\mu((xa)y)\geq\mu(a))$.

Let μ be a fuzzy subset of G and $\phi\neq A\subseteq G$ then the *characteristic function* μ_A of A is defined as:

$$\mu_A : G\to[0,1], a\mapsto\mu_A(a):=\begin{cases}1 \text{ if } a\in A,\\ 0 \text{ if } a\notin A.\end{cases}$$

Let μ and λ be the two fuzzy subsets of AG-groupoid G. Then the product $\mu\circ\lambda$ is defined by

$$(\mu\circ\lambda)(x)=\begin{cases}\bigvee_{x=yz}(\mu(y)\wedge\lambda(z)) & \text{if } x=yz,\\ 0 & \text{otherwise}\end{cases}$$

for all $y,z\in G$.

For an AG-groupoid G, the fuzzy subsets "0" and "1" of G are defined as follows:

$$0:G\to[0,1]\,|\,x\to 0(x)=0,$$
$$1:G\to[0,1]\,|\,x\to 1(x)=1.$$

Clearly, the fuzzy subset 0 (resp. 1) of G is the least (resp. the greatest) element of the AG-groupoid $(\mu(G),\circ)$ (that is, $0\leq\mu$ and $\mu\leq 1$ for every $\mu\in\mu(G)$). The fuzzy subset 0 is the zero element of $(\mu(G),\circ)$ (that is, $\mu\circ 0=0\circ\mu=0$ and $0\leq\mu$ for every $\mu\in(\mu(G))$). Moreover, $\mu_S=1$ and $\mu_\phi=0$.

2.1. Lemma (cf. [23]).

Let G be an AG-groupoid and μ a fuzzy subset of G. Then μ is a fuzzy bi-ideal of G if and only if μ_A is a fuzzy bi-ideal of G.

Let G be an AG-groupoid and μ a fuzzy subset of G. Then for every $t\in(0,1]$ the set

$U(\mu;t):=\{x\,|\,x\in G \text{ and } \mu(x)\geq t\}$

is called a *level set* of μ with support x and value t.

2.2. Theorem (cf. [23]).

Let G be an AG-groupoid and μ a fuzzy subset of G. Then μ is a fuzzy bi-ideal of G if and only if $U(\mu;t)(\neq\phi)$ is a bi-ideal of G for every $t\in(0,1]$.

2.3. Theorem (cf. [24]).

Let G be an AG-groupoid and μ a fuzzy subset of G. Then μ is a fuzzy interior ideal of G if and only if $U(\mu;t)(\neq\phi)$ is an interior ideal of G for every $t\in(0,1]$.

Let μ be a fuzzy subset of G, then the set of the form

$$\mu(y):=\begin{cases}t(\neq 0) & \text{if } y=x,\\ 0 & \text{otherwise,}\end{cases}$$

is called a *fuzzy point* with support x and value t and is denoted by $[x;t]$. A fuzzy point $[x;t]$ is said to *belong to* (resp. *quasi-coincidence*) with a fuzzy set μ, written as $[x;t]\in\mu$ (resp. $[x;t]q\mu$) if $\mu(x)\geq\lambda$ (resp. $\mu(x)+\lambda>1$). If $[x;t]\in\mu$ or $[x;t]q\mu$, then we write $[x;t]\in\vee q\mu$. The symbol $\overline{\in\vee q}$ means $\in\vee q$ does not hold.

2.4. Definition

A fuzzy subset μ of G is called an *$(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal* of G if it satisfies the following conditions:

(C1)

$(\forall x,y\in G)(\forall s,t\in(0,1])([xy;s\wedge t]\overline{\in}\mu\Rightarrow[x;s]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu)$,

(C2)

$(\forall x,a,y\in G)(\forall s,t\in(0,1])([(xa)y;s\wedge t]\overline{\in}\mu\Rightarrow[x;s]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu)$.

2.5. Example

Consider $G=\{a,b,c,d,e\}$ with the following multiplication table:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	e	c	d
d	a	a	d	e	c
e	a	a	c	d	e

Then $(G,\cdot)$ is an AG-groupoid. Define a fuzzy subset $\mu : G\to[0,1]$ as follows:

$$\mu(x)=\begin{cases}0.8 & \text{if } x=a,\\ 0.6 & \text{if } x\in\{b,c,d,e\}.\end{cases}$$

Then by Definition 2.4, μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of G.

2.6. Theorem

If μ is a fuzzy subset of G, then μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy bi-ideal of G if and only if

$(C3)$ $(\forall x,y\in G)(\mu(xy)\vee 0.5\geq \mu(x)\wedge\mu(y))$,

$(C4)(\forall x,a,y\in G)(\mu((xa)y)\vee 0.5\geq \mu(x)\wedge\mu(y))$.

Proof $(C1)\Rightarrow(C3)$. If there exist $x,y\in G$ such that $\mu(xy)\vee 0.5<t=\mu(x)\wedge\mu(y)$,

then $0.5<t\leq 1, [xy;t]\overline{\in}\mu$ but $[x;t]\in\mu$ and $[y;t]\in\mu$. By $(C1)$, we have $[x;t]\overline{q}\mu$ or $[y;t]\overline{q}\mu$. Then ($\mu(x)\geq t$ and $t+\mu(x)\leq 1$) or ($\mu(y)\geq t$ and $t+\mu(y)\leq 1$), which implies that $t\leq 0.5$, a contradiction.

$(C3)\Rightarrow(C1)$. Let $x,y\in G$ and $s,t\in(0,1]$ be such that $[xy;s\wedge t]\overline{\in}\mu$, then $\mu(xy)<s\wedge t$.

(a) If $\mu(xy)\geq\mu(x)\wedge\mu(y)$, then $\mu(x)\wedge\mu(y)<s\wedge t$, and consequently, $\mu(x)<s$ or $\mu(y)<t$. It follows that $[x;s]\overline{\in}\mu$ or $[y;t]\overline{\in}\mu$. Thus $[x;s]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu$.

(b) If $\mu(xy)<\mu(x)\wedge\mu(y)$, then by $(C3)$, we have $0.5\geq\mu(x)\wedge\mu(y)$. Let $[x;s]\overline{\in}\mu$ or $[y;t]\overline{\in}\mu$, then $s\leq\mu(x)\leq 0.5$ or $t\leq\mu(y)\leq 0.5$. It follows $[x;s]\overline{q}\mu$ or $[y;t]\overline{q}\mu$, and $[x;s]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu$.

(C2) $\Rightarrow$ (C4). If there exist $x,a,y\in G$ such that $\mu((xa)y)\vee 0.5<t=\mu(x)\wedge\mu(y)$, then $0.5<t\leq 1,[(xa)y;t]\overline{\in}\mu$ but $[x;t]\in\mu$ and $[y;t]\in\mu$. By (C2), we have $[x;t]\overline{q}\mu$ or $[y;t]\overline{q}\mu$. Then ($\mu(x)\geq t$ and $t+\mu(x)\leq 1$) or ($\mu(y)\geq t$ and $t+\mu(y)\leq 1$), which implies that $t\leq 0.5$, a contradiction.

(C4)$\Rightarrow$ (C2). Let $x,a,y\in G$ and $r,t\in(0,1]$ be such that $[(xa)y;r\wedge t]\overline{\in}\mu$, then $\mu((xa)y)<r\wedge t$.

(a) If $\mu((xa)y)\geq\mu(x)\wedge\mu(y)$, then $\mu(x)\wedge\mu(y)<r\wedge t$, and consequently, $\mu(x)<r$ or $\mu(y)<t$. It follows that $[x;r]\overline{\in}\mu$ or $[y;t]\overline{\in}\mu$. Thus $[x;r]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu$.

(b) If $\mu((xa)y)<\mu(x)\wedge\mu(y)$, then by (C4), we have $0.5\geq\mu(x)\wedge\mu(y)$. Let $[x;r]\overline{\in}\mu$ or $[y;t]\overline{\in}\mu$, then $r\leq\mu(x)\leq 0.5$ or $t\leq\mu(y)\leq 0.5$. It follows $[x;r]\overline{q}\mu$ or $[y;t]\overline{q}\mu$, and $[x;r]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu$.

2.7. Definition

A fuzzy subset μ of G is called an *$(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior ideal* of G if it satisfies the following conditions:

$(I1)$

$(\forall x,y\in G)(\forall s,t\in(0,1])([xy;s\wedge t]\overline{\in}\mu\Rightarrow[x;s]\overline{\in}\vee\overline{q}\mu$ or $[y;t]\overline{\in}\vee\overline{q}\mu)$,

$(I2)$

$(\forall x,a,y\in G)(\forall s\in(0,1])([(xa)y;s]\overline{\in}\mu\Rightarrow[a;s]\overline{\in}\vee\overline{q}\mu$.

2.8. Theorem

If μ is a fuzzy subset of G, then μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy interior ideal of G if and only if

$(I3)$ $(\forall x,y\in G)(\mu(xy)\vee 0.5\geq\mu(x)\wedge\mu(y))$,

$(I4)(\forall x,a,y\in G)(\mu((xa)y)\vee 0.5\geq\mu(a))$.

Proof $(I2)\Rightarrow(I4)$. If there exist $x,a,y\in G$ such that $\mu((xa)y)\vee 0.5<t=\mu(a)$,

then $0.5<t\leq 1,[(xa)y;t]\overline{\in}\mu$ but $[a;t]\in\mu$. By $(I2)$, we have $[a;t]\overline{q}\mu$. Then ($\mu(a)\geq t$ and $t+\mu(a)\leq 1$) which implies that $t\leq 0.5$, a contradiction.

$(I4)\Rightarrow(I2)$. Let $x,a,y\in G$ and $r\in(0,1]$ be such that $[(xa)y;r]\overline{\in}\mu$, then $\mu((xa)y)<r$.

(a) If $\mu((xa)y)\geq\mu(a)$, then $\mu(a)<r$, and consequently, $\mu(a)<r$. It follows that $[a;r]\overline{\in}\mu$. Thus $[a;r]\overline{\in}\vee\overline{q}\mu$.

(b) If $\mu((xa)y)<\mu(a)$, then by $(I4)$, we have $0.5\geq\mu(a)$. Let $[a;r]\overline{\in}\mu$, then $r\leq\mu(a)\leq 0.5$. It follows $[a;r]\overline{q}\mu$, so $[a;r]\overline{\in}\vee\overline{q}\mu$. The remaining proof follows from Theorem 2.6.

2.9. Definition

A fuzzy subset μ of G is called an *$(\overline{\in},\overline{\in}\vee\overline{q})$-fuzzy left* (resp. *right*) *ideal* of G if it satisfies the following conditions:

$(\forall x,y\in G)(\forall s\in(0,1])([xy;s]\overline{\in}\mu\Rightarrow[y;s]\overline{\in}\vee\overline{q}\mu$

(resp. $[x;s]\overline{\in}\vee\overline{q}\mu$).

2.10. Theorem

If μ is a fuzzy subset of G, then μ is an $(\overline{\in},\overline{\in}\vee\overline{q})$-*fuzzy left* (resp. *right*) *ideal* of G if and only if

$(\forall x,y\in G)(\mu((xy)\vee 0.5\geq\mu(y)(\text{resp. }\mu(x)))$.

Proof. Suppose that μ is an $(\bar{\in},\bar{\in}\vee\bar{q})$ -fuzzy left ideal. If there exist $x,y\in G$ such that $\mu(xy)\vee 0.5<t=\mu(y)$, then $0.5<t\leq 1,[xy;t]\bar{\in}\mu$ but $[y;t]\in\mu$. By Definition 2.9, we have $[a;t]\bar{q}\mu$. Then ($\mu(a)\geq t$ and $t+\mu(a)\leq 1$) which implies that $t\leq 0.5$, a contradiction. Let $x,y\in G$ and $s\in(0,1]$ be such that $[xy;s]\bar{\in}\mu$, then $\mu(xy)<s$.

(a) If $\mu(xy)\geq\mu(y)$, then $\mu(y)<s$. It follows that $[y;s]\bar{\in}\mu$. Thus $[y;s]\bar{\in}\vee\bar{q}\mu$.

(b) If $\mu(xy)<\mu(y)$, then by hypothesis, we have $0.5\geq\mu(y)$. Let $[y;s]\bar{\in}\mu$, then $s\leq\mu(y)\leq 0.5$. It follows $[y;s]\bar{q}\mu$, so $[y;a]\bar{\in}\vee\bar{q}\mu$. By similar way the case for right ideal can be proved.

3. $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideals of AG -groupoids

Ma et al. [27] investigated new types of fuzzy ideals called $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy (positive implicative, implicative and commutative) ideals and $(\bar{\in}_\gamma,\bar{\in}_\gamma\vee\bar{q}_\delta)$ -fuzzy (positive implicative, implicative and commutative) ideals of BCI -algebras and provided the classification of BCI -algebras in terms of these types of fuzzy ideals. The concept of general forms of a fuzzy interior ideal in an AG-groupoid was studied in [24]. Generalizing the concepts of [23,24], here we extend our studies to more general forms of fuzzy bi-deals, fuzzy left (right) ideals and fuzzy interior ideals in AG -groupoids. In this section, some new types of fuzzy bi-ideals called (α,β) -fuzzy bi-ideals of G are introduced and investigate some new types of characterization of AG-groupoids.

In what follows, let $\gamma,\delta\in[0,1]$ be such that $\gamma<\delta$. For a fuzzy point $[x;s]$ and a fuzzy subset μ of X, we say that

(1) $[x;r]\in_\gamma\mu$ if $\mu(x)\geq r>\gamma$.

(2) $[x;r]q_\delta\mu$ if $\mu(x)+r>2\delta$.

(3) $[x;r]\in_\gamma\vee q_\delta\mu$ if $[x;r]\in_\gamma\mu$ or $[x;r]q_\delta\mu$.

(4) $[x;r]\in_\gamma\wedge q_\delta\mu$ if $[x;r]\in_\gamma\mu$ and $[x;r]q_\delta\mu$.

(5) $[x;r]\bar{\alpha}\mu$ if $[x;r]\alpha\mu$ does not hold for $\alpha\in\{\in_\gamma,q_\delta,\in_\gamma\vee q_\delta,\in_\gamma\wedge q_\delta\}$.

3.1. Definition

A fuzzy subset μ of G is called an (α,β) *-fuzzy AG -subgroupoid* of G if it satisfies the conditions:

$(\forall x,y\in G)(\forall s,t\in(\gamma,1])$

$([x;s]\alpha\mu,[x;t]\alpha\mu\rightarrow[xy;s\wedge t]\beta\mu)$.

Note that $\alpha,\beta\in\{\in_\gamma,q_\delta,\in_\gamma\vee q_\delta,\in_\gamma\wedge q_\delta\}$ and $\alpha\neq\in_\gamma\wedge q_\delta$. The case $\alpha=\in_\gamma\wedge q_\delta$ is omitted. Because for a fuzzy subset μ such that $\mu(x)<\delta$ for any $x\in G$. In the case $[x;t]\in_\gamma\wedge q_\delta\mu$ we have $\mu(x)\geq t$ and $\mu(x)+t>2\delta$. Therefore $\mu(x)+\mu(x)>\mu(x)+t>2\delta$ which implies that $2\mu(x)>2\delta$. Thus $\mu(x)>\delta$, a contradiction. This means that $\{[x;t]\mid[x;t]\in_\gamma\wedge q_\delta\mu\}=\phi$.

3.2. Theorem

A fuzzy subset μ of G is an $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy AG-subgroupoid of G if and only if

$(\forall x,y\in G)(\gamma,\delta\in[0,1])(\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta)$.

Proof. Assume that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$ -fuzzy AG-subgroupoid of G. If there exist $x,y\in G$ such that $\mu(xy)\vee\gamma<\mu(x)\wedge\mu(y)\wedge\delta$. Choose $t\in(\gamma,1]$ such that $\mu(xy)\vee\gamma<t\leq\mu(x)\wedge\mu(y)\wedge\delta$. Then $[x;t]\in_\gamma\mu$ and $[y;t]\in_\gamma\mu$ but $\mu(xy)<t$ and $\mu(xy)+t<2t<2\delta$, so $[xy;t]\overline{\in_\gamma\vee q_\delta}\mu$ a contradiction. Hence $\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,y\in G$ and $\gamma,\delta\in[0,1]$.

Conversely, assume that $\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,y\in G$ and $\gamma,\delta\in[0,1]$. Let $[x;s]\in_\gamma\mu$ and $[y;t]\in_\gamma\mu$ for some $s,t\in(\gamma,1]$, then $\mu(x)\geq s>\gamma$ and $\mu(y)\geq t>\gamma$ and by hypothesis,

$$\begin{aligned}\mu(xy)\vee\gamma&\geq\mu(x)\wedge\mu(y)\wedge\delta\\&\geq s\wedge t\wedge\delta\\&=\begin{cases}s\wedge t & \text{if } s\wedge t\leq\delta\\ \delta & \text{if } \delta<s\wedge t\end{cases}.\end{aligned}$$

That is $\mu(xy)\vee\gamma\geq s\wedge t$, but $s\wedge t>\gamma$ therefore $\mu(xy)\geq s\wedge t>\gamma$ and hence $[xy;s\wedge t]\in_\gamma\mu$ or $\mu(xy)\vee\gamma\geq\delta$ but $\gamma<\delta$, therefore $\mu(xy)\geq\delta$, thus $\mu(xy)+s\wedge t\geq\delta+s\wedge t>2\delta$ that is $[xy;s\wedge t]q_\delta\mu$ hence $[xy;s\wedge t]\in_\gamma\vee q_\delta\mu$. Consequently μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy AG-subgroupoid of G.

3.3. Definition

A fuzzy subset μ of G is called an (α,β) *-fuzzy bi-ideal* of G if it satisfies the conditions:

(1) $\left(\begin{matrix}\forall x,y\in G)(\forall s,t\in(\gamma,1])\\ ([x;s]\alpha\mu,[x;t]\alpha\mu\rightarrow[xy;s\wedge t]\beta\mu\end{matrix}\right)$,

(2) $\left(\begin{matrix}\forall x,a,y\in G)(\forall s,t\in(\gamma,1])\\ ([x;s]\alpha\mu,[y;t]\alpha\mu\rightarrow[(xa)y;s\wedge t]\beta\mu\end{matrix}\right)$.

3.4. Example

Let G be an AG-groupoid as shown in Example 2.5. Then $\{a\},\{a,c,d,e\}$ and G are bi-ideals of G. Define a fuzzy subset $\mu : S\rightarrow[0,1]$ as follows:

$$\mu(x)=\begin{cases}0.8 & \text{if } x=a,\\ 0.6 & \text{if } x=b,\\ 0.2 & \text{if } x=c,\\ 0.3 & \text{if } x=d,\\ 0.4 & \text{if } x=e.\end{cases}$$

Then μ is an $(\in_{0.1},\in_{0.1}\vee q_{0.2})$-fuzzy bi-ideal of G.

3.5. Theorem

Let μ be a fuzzy subset of G. Then the following conditions are equivalent:

(1) μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal of G.

(2) μ satisfies the conditions:

$(2.1)\left(\begin{matrix}(\forall x,y\in G)(\forall\gamma,\delta\in[0,1])\\ (\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta)\end{matrix}\right)$,

$(2.2)\left(\begin{matrix}(\forall x,a,y\in G)(\forall\gamma,\delta\in[0,1))\\ (\mu((xa)y)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta)\end{matrix}\right)$.

Proof Let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal of G and consider $(\mu(xy)\vee\gamma<\mu(x)\wedge\mu(y)\wedge\delta)$ for some $x,y\in G$ and $\gamma,\delta\in[0,1]$, then $(\mu(xy)\vee\gamma<t\leq\mu(x)\wedge\mu(y)\wedge\delta)$ for some $t\in(\gamma,1]$. From this we observe that $[x;t]\in_\gamma\mu$ and $[y;t]\in_\gamma\mu$ but $\mu(xy)<t$ and $\mu(xy)+t<2t<2\delta$, so $[xy;t]\overline{\in_\gamma\vee q_\delta}\mu$ a contradiction. Hence $\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,y\in G$ and $\gamma,\delta\in[0,1]$. Next, $\mu((xa)y)\vee\gamma<\mu(x)\wedge\mu(y)\wedge\delta$ for some $x,a,y\in G$ and $\gamma,\delta\in[0,1]$, then there exist some $t\in(\gamma,1]$ such that $\mu((xa)y)\vee\gamma<t\leq\mu(x)\wedge\mu(y)\wedge\delta$. This shows that $[x;t]\in_\gamma\mu$ and $[y;t]\in_\gamma\mu$ but $\mu((xa)y)<t$ and $\mu((xa)y)+t<2t<2\delta$, so $[(xa)y;\lambda]\overline{\in_\gamma\vee q_\delta}\mu$ a contradiction. Therefore, $\mu((xa)y)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,a,y\in G$ and $\gamma,\delta\in[0,1]$.

Conversely; Let μ satisfies conditions (2.1) and (2.2) and consider $[x;s]\in_\gamma\mu,[y;t]\in_\gamma\mu$ for some $x,y\in G$ and $s,t\in(\gamma,1]$. Then by (2.1)

$$\begin{aligned}\mu(xy)\vee\gamma&\geq\mu(x)\wedge\mu(y)\wedge\delta\\ &\geq s\wedge t\wedge\delta\\ &=\begin{cases}s\wedge t & \text{if } s\wedge t\leq\delta\\ \delta & \text{if } \delta<s\wedge t\end{cases}.\end{aligned}$$

This shows that $[xy;s\wedge t]\in_\gamma\vee q_\delta\mu$. And by (2.2),

$$\begin{aligned}\mu((xa)y)\vee\gamma&\geq\mu(x)\wedge\mu(y)\wedge\delta\\ &\geq s\wedge t\wedge\delta\\ &=\begin{cases}s\wedge t & \text{if } s\wedge t\leq\delta\\ \delta & \text{if } \delta<s\wedge t\end{cases}.\end{aligned}$$

Follows from Theorem 3.2, $[(xa)y;s\wedge t]\in_\gamma\vee q_\delta\mu$. Consequently μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy bi-ideal of G.

3.6. Proposition

Let $2\delta=1+\gamma$ and A be a non-empty subset of G. Then A is an AG-subgroupoid of G if and only if the fuzzy subset μ of G defined as

$$\mu(x)=\begin{cases}\geq\delta & \text{if } x\in A,\\ \gamma & \text{if } x\in G\setminus A,\end{cases}$$

is an $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy AG-subgroupoid of G.

Proof The proof follows from [28].

3.7. Theorem

Let μ be a fuzzy subset of G. Then the following conditions are equivalent:

(1) μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy AG-subgroupoid of G.

(2) $U(\mu;t)(\neq\phi)$ is an AG-subgroupoid of G for all $t\in(\gamma,\delta]$.

Proof Suppose that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy AG-subgroupoid of G and $x,y\in U(\mu;t)$ for some $t\in(\gamma,\delta]$. Then $\mu(x)\geq t$ and $\mu(y)\geq t$ and by hypothesis

$$\begin{aligned}\mu(xy)\vee\gamma&\geq\mu(x)\wedge\mu(y)\wedge\delta\\ &\geq t\wedge t\wedge\delta\\ &=t\ (\text{since } t\in(\gamma,\delta]).\end{aligned}$$

Hence $\mu(xy)\geq t>\gamma$ therefore, $\mu(xy)\in U(\mu;t)$. Thus $U(\mu;t)$ is an AG-subgroupoid of G.

Conversely, assume that $U(\mu;t)(\neq\phi)$ is an AG-subgroupoid of G for all $t\in(\gamma,\delta]$. If there exit $x,y\in G$ such that $\mu(xy)\vee\gamma<\mu(x)\wedge\mu(y)\wedge\delta$. Then $\mu(xy)\vee\gamma<t\leq\mu(x)\wedge\mu(y)\wedge\delta$ for some $t\in(\gamma,1]$. From this we see that $x,y\in U(\mu;t)$ but $xy\notin U(\mu;t)$, a contradiction. Hence $\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,y\in G$ and $\gamma,\delta\in[0,1]$.

The proof of Proposition 3.8, is straightforward and is omitted.

3.8. Proposition

Let $2\delta = 1+\gamma$ and A be a non-empty subset of G. Then A is a bi-ideal of G if and only if the fuzzy subset μ of G defined as

$$\mu(x) = \begin{cases} \geq \delta \text{ if } x \in A, \\ \gamma \text{ if } x \in G \setminus A, \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof The proof follows from [28].

3.9. Theorem

Let μ be a fuzzy subset of G. Then the following conditions are equivalent:

(1) μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

(2) $U(\mu;t)(\neq \phi)$ is a bi-ideal of G for all $t \in (\gamma, \delta]$.

Proof. Suppose that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G and let $x, a, y \in G$ be such that $x, y \in U(\mu;t)$ for some $t \in (\gamma, \delta]$. Then $\mu(x) \geq t$ and $\mu(y) \geq t$ and by hypothesis

$$\begin{aligned} \mu((xa)y) \vee \gamma &\geq \mu(x) \wedge \mu(y) \wedge \delta) \\ &\geq t \wedge t \wedge \delta \\ &= t \text{ (since } t \in (\gamma, \delta]). \end{aligned}$$

Hence $\mu((xa)y) \geq t > \gamma$, this shows that $(xa)y \in U(\mu;t)$ and hence $U(\mu;t)(\neq \phi)$ is a bi-ideal of G for all $x, a, y \in G$ and $t \in (\gamma, \delta]$.

Conversely, assume that $U(\mu;t)(\neq \phi)$ is a bi-ideal of G for all $t \in (\gamma, \delta]$. If there exist $x, a, y \in G$ such that $\mu((xa)y) \vee \gamma < \mu(x) \wedge \mu(y) \wedge \delta)$. Then choose $t \in (\gamma, \delta]$ such that $\mu((xa)y) \vee \gamma < t \leq \mu(x) \wedge \mu(y) \wedge \delta)$. From here we have $x, y \in U(\mu;t)$ but ($xa)y \notin U(\mu;t)$, a contradiction. Hence $\mu((xa)y) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$ for all $x, a, y \in G$ and $t \in (\gamma, \delta]$. The remaining proof is a consequence of the Theorem 3.7. Therefore μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

3.10. Lemma

A non-empty subset A of G is a bi-ideal if and only if the characteristic function μ_A of A is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof The proof is straightforward.

3.11. Definition

A fuzzy subset μ of G is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-*fuzzy left* (resp. *right*) *ideal* of G if the following condition holds:

$(\forall x, y \in S)(\forall \gamma, \delta \in [0,1])$

$([y;t] \in_\gamma \mu \rightarrow [xy;t] \in_\gamma \vee q_\delta \mu (\text{resp. } [yx;t] \in_\gamma \vee q_\delta \mu)$.

3.12. Theorem

Let $2\delta = 1+\gamma$ and A be a non-empty subset of G. Then A is a left (resp. right) ideal of G if and only if the fuzzy subset μ of G defined as

$$\mu(x) = \begin{cases} \geq \delta \text{ if } x \in A, \\ \gamma \text{ if } x \in G \setminus A, \end{cases}$$

is an $(\alpha, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right) ideal of G.

Proof To prove this, we consider the following three cases:

Case 1. Consider A is a left ideal of G. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y;t] \in_\gamma \mu$, then $y \in A$. Since A is a left ideal of G, therefore $xy \in A$ that is $\mu(xy) \geq \delta$. If $t \leq \delta$, then $\mu(xy) \geq t > \gamma$ and so $[xy;t] \in_\gamma \mu$. If $t > \delta$, then $\mu(xy) + t > \delta + t > 2\delta$ and hence $[xy;t] q_\delta \mu$. Thus $[xy;t] \in_\gamma \vee q_\delta \mu$. This shows that μ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Case 2. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y;t] q_\delta \mu$, then $y \in A$. Therefore $xy \in A$, since A is a left ideal of G that is $\mu(xy) \geq \delta$. If $t \leq \delta$, then $\mu(xy) \geq t > \gamma$ and so $[xy;t] \in_\gamma \mu$. If $t > \delta$, then $\mu(xy) + t > \delta + t > 2\delta$ and hence $[xy;t] q_\delta \mu$. Thus $[xy;t] \in_\gamma \vee q_\delta \mu$. This shows that μ is an $(q_\delta, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G.

Case 3. Let $x, y \in G$ and $t \in (\gamma, 1]$ be such that $[y;t] \in_\gamma \vee q_\delta \mu$, follows from Case 1 and Case 2.

3.13. Proposition

Every $(\in_\gamma, \in_\gamma)$-fuzzy bi-ideal of G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof It is straightforward, since for $[x;t] \in_\gamma \mu$ we have $[x;t] \in_\gamma \vee q_\delta \mu$ for all $x \in G$ and $t \in (\gamma, 1]$.

The converse of Proposition 3.13, is not true in general, as shown in the following example:

3.14. Example

Consider an AG-groupoid G and a fuzzy subset μ as defined in Example 2.5, then μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy bi-ideal of G but μ is not an

$(\in_{0.1}, \in_{0.1})$ -fuzzy bi-ideal of G, since $[d;0.28] \in_{0.1} \mu$ $[e;0.38] \in_{0.1} \mu$ and but $[(dc)e; 0.28 \wedge 0.38] = [c;0.28] \overline{\in}_{0.1} \mu.$

3.15. Remark

A fuzzy subset μ of an AG-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G if and only if it satisfies conditions (2.1), and (2.2) of Theorem 3.5.

3.16. Remark

Every fuzzy bi-ideal of an AG-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideal of G. However, the converse is not true in general.

3.17. Example

Consider the AG-groupoid as given in Example 2.5, and define a fuzzy subset μ as follows:

$\mu(a) = 0.8, \ \ \mu(b) = 0.6, \ \ \mu(c) = 0.2, \ \ \mu(d) = 0.3, \ \ \mu(e) = 0.4.$

Then μ is an $(\in_{0.1}, \in_{0.1} \vee q_{0.2})$-fuzzy bi-ideal of G. However,

i) μ is not an $(\in_{0.1}, q_{0.2})$-fuzzy bi-ideal of G, since $[d;0.18] \in_{0.1} \mu$ and $[e;0.20] \in_{0.1} \mu$ but $[(dc)e; 0.18 \wedge 0.20] = [c;0.18] \overline{q}_{0.2} \mu$

ii) μ is not an $(q_{0.2}, \in_{0.1})$-fuzzy bi-ideal of G, since $[c;0.28] \in_{0.1} \mu$ and $[d;0.38] \in_{0.1} \mu$ but $[(cd); 0.28 \wedge 0.38] = [c;0.28] \overline{\in}_{0.1} \mu$

iii) μ is not an $(\in_{0.1} \vee q_{0.2}, q_{0.2})$ -fuzzy bi-ideal of G, since $[e;0.18] \in_{0.1} \vee q_{0.2} \mu$ and $[d;0.28] \in_{0.1} \vee q_{0.2} \mu$ but $[(ec); 0.18 \wedge 0.20] = [c;0.18] \overline{q}_{0.2} \mu$

Thus μ is not an (α, β)-fuzzy bi-ideal of G, for every $\alpha, \beta \in \{\in_{0.1}, q_{0.2}, \in_{0.1} \vee q_{0.2}\}$.

3.18. Proposition

If $\{\mu_i\}_{i \in I}$ is a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of an AG-groupoid G, then $\bigcap_{i \in I} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof. Let $\{\mu_i\}_{i \in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideals of G. Let $x, y \in G$. Then

$$\left(\bigcap_{i \in I} \mu_i\right)(xy) \vee \gamma = \bigwedge_{i \in I} \mu_i(xy) \vee \gamma \geq \bigwedge_{i \in I} (\mu_i(x) \wedge \mu_i(y) \wedge \delta)$$
$$= \left(\bigwedge_{i \in I} (\mu_i(x) \wedge \delta) \wedge \bigwedge_{i \in I} (\mu_i(y) \wedge \delta)\right)$$
$$= \left(\bigcap_{i \in I} \mu_i\right)(x) \wedge \left(\bigcap_{i \in I} \mu_i\right)(y) \wedge \delta.$$

Let $x, a, y \in G$. Then

$$\left(\bigcap_{i \in I} \mu_i\right)((xa)y) \vee \gamma = \bigwedge_{i \in I} \mu_i((xa)y) \vee \gamma \geq \bigwedge_{i \in I} (\mu_i(x) \wedge \mu_i(y) \wedge \delta)$$
$$= \left(\bigwedge_{i \in I} (\mu_i(x) \wedge \delta) \wedge \bigwedge_{i \in I} (\mu_i(y) \wedge \delta)\right)$$
$$= \left(\bigcap_{i \in I} \mu_i\right)(x) \wedge \left(\bigcap_{i \in I} \mu_i\right)(y) \wedge \delta.$$

Thus $\bigcap_{i \in I} \mu_i$ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

3.19. Lemma

The intersection of any family $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right) ideals of an AG-groupoid G is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left (resp. right) ideal of G.

Proof. We consider the case for a left ideal. The case for a right ideal can be proved similarly. Let $\{\mu_i\}_{i \in I}$ be a family of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideals of G and $x, y \in G$. Then $\left(\bigwedge_{i \in I} \mu_i\right)(xy) = \bigwedge_{i \in I} (\mu_i(xy))$. Since each μ_i is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy left ideal of G, we have $\mu_i(xy) \vee \gamma \geq \mu_i(y) \wedge \delta$ for all $i \in I$. Thus

$$\left(\bigwedge_{i \in I} \mu_i\right)(xy) \vee \gamma = \bigwedge_{i \in I} \left(\mu_i(xy) \vee \gamma\right)$$
$$\geq \bigwedge_{i \in I} \left(\mu_i(y) \wedge \delta\right)$$
$$= \left(\bigwedge_{i \in I} \mu_i(y)\right) \wedge \delta$$
$$= \left(\bigwedge_{i \in I} \mu_i\right)(y) \wedge \delta.$$

Hence $\bigwedge_{i\in I}\mu_i$ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left ideal of G.

3.20. Lemma

The union of any family $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left (resp. right) ideals of an AG-groupoid G is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy left (resp. right) ideal of G.

4. $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideals

In this section, some new types of fuzzy interior ideals of AG-groupoids are introduced. An important milestone of this section is to provide a connection between ordinary interior ideals and fuzzy interior ideals of type $(\in_\gamma,\in_\gamma\vee q_\delta)$ of AG-groupoids.

4.1. Definition

A fuzzy subset μ of an AG-groupoid G is called an $(\in_\gamma,\in_\gamma\vee q_\delta)$ *-fuzzy interior ideal* of G if it satisfies the following conditions:

$(I1)(\forall x,y\in G)(\forall s,t\in(\gamma,1])$

$([x;s]\in_\gamma\mu,[y;t]\in_\gamma\mu\rightarrow[xy;s\wedge t]\in_\gamma\vee q_\delta\mu),$

$(I2)(\forall x,a,y\in G)(\forall s,t\in(\gamma,1])$

$([a;s]\in_\gamma\mu\rightarrow[(xa)y;s\wedge t]\in_\gamma\vee q_\delta\mu).$

4.2. Theorem

Let $2\delta=1+\gamma$ and A be a non-empty subset of G. Then A is an interior ideal of G if and only if the fuzzy subset μ of G defined as

$$\mu(x)=\begin{cases}\geq\delta \text{ if } x\in A,\\ \gamma \text{ if } x\in G\setminus A,\end{cases}$$

is an $(\alpha,\in_\gamma\vee q_\delta)$-fuzzy interior ideal of G.

Proof. We consider the following three cases:

Case 1: Consider A is an interior ideal of G. Let $x,a,y\in G$ and $[a;t]\in_\gamma\mu$ for some $t\in(\gamma,1]$, then $a\in A$. Since A is an interior ideal of G, therefore $xay\in A$ that is $\mu(xay)\geq\delta$. If $t\leq\delta$, then $\mu(xay)\geq t$ and so $[xay;t]\in_\gamma\mu$. If $t>\delta$, then $\mu(xay)+t>\delta+t>2\delta$ and hence $[xay;t]q_\delta\mu$. Thus $[xay;t]\in_\gamma\vee q_\delta\mu$. This shows that μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideal of G.

Case 2: Let $x,a,y\in G$ and $t\in(\gamma,1]$ such that $[a;t]q_\delta\mu$, then $a\in A$. Since A is an interior ideal of G, therefore $xay\in A$ hence $\mu(xay)\geq\delta$. If $t\leq\delta$, then $\mu(xay)\geq t$ and so $[xay;t]\in_\gamma\mu$. If $t>\delta$, then $\mu(xay)+t>\delta+t>2\delta$ and hence $[xay;t]q_\delta\mu$. Thus $[xay;t]\in_\gamma\vee q_\delta\mu$. This shows that μ is an $(q_\delta,\in_\gamma\vee q_\delta)$-fuzzy left ideal of G.

Case 3. Let $x,a,y\in G$ and $t\in(\gamma,1]$ be such that $[a;t]\in_\gamma\vee q_\delta\mu,$ follows from Case 1 and Case 2. The remaining proof follows directly from Proposition 3.8.

4.3. Lemma

A non-empty subset A of G is an interior ideal if and only if the characteristic function μ_A of A is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideal of G.

Proof The proof follows from Lemma 3.10.

4.4. Theorem

Suppose that μ is a fuzzy subset of G. Then the following conditions are equivalent:

(1) μ is an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideal of G.

(2) μ satisfies the conditions:

$(2.1)(\forall x,y\in G)(\forall\gamma,\delta\in[0,1])$
$(\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta)$,

$(2.2)(\forall x,a,y\in G)(\forall\gamma,\delta\in[0,1))$
$(\mu((xa)y)\vee\gamma\geq\mu(a)\wedge\delta)$.

Proof. Let μ be an $(\in_\gamma,\in_\gamma\vee q_\delta)$-fuzzy interior ideal of G. Suppose that $(\mu(xy)\vee\gamma<\mu(x)\wedge\mu(y)\wedge\delta)$ for some $x,y\in G$ and $\gamma,\delta\in[0,1]$, then $(\mu(xy)\vee\gamma<t\leq\mu(x)\wedge\mu(y)\wedge\delta)$ for some $t\in(\gamma,1]$. From this we observe that $[x;t]\in_\gamma\mu$ and $[y;t]\in_\gamma\mu$ but $\mu(xy)<t$ and $\mu(xy)+t<2t\leq 2\delta$, so $[xy;t]\overline{\in_\gamma\vee q_\delta}\mu$ a contradiction. Hence $\mu(xy)\vee\gamma\geq\mu(x)\wedge\mu(y)\wedge\delta$ for all $x,y\in G$ and $\gamma,\delta\in[0,1]$. Next, $\mu((xa)y)\vee\gamma<\mu(a)\wedge\delta$ for some $x,a,y\in G$ and $\gamma,\delta\in[0,1]$, then there exist some $t\in(\gamma,1]$ such that $\mu((xa)y)\vee\gamma<t\leq\mu(a)\wedge\delta$. This shows that $[a;t]\in_\gamma\mu$ but $\mu((xa)y)<t$ and $\mu((xa)y)+t<2t\leq2\delta$, so $[(xa)y;t]\overline{\in_\gamma\vee q_\delta}\mu$ a contradiction. This implies $\mu((xa)y)\vee\gamma\geq\mu(a)\wedge\delta$ for all $x,a,y\in G$ and $\gamma,\delta\in[0,1]$.

Conversely; let μ satisfies conditions (2.1) and (2.2) and consider $[x;s]\in_\gamma\mu,[y;t]\in_\gamma\mu$ for some $x,y\in G$ and $s,t\in(\gamma,1]$. Then by (2.1)

$$\begin{aligned}\mu(xy)\vee\gamma&\geq\mu(x)\wedge\mu(y)\wedge\delta\\&\geq s\wedge t\wedge\delta\\&=\begin{cases}s\wedge t \text{ if } s\wedge t\leq\delta\\ \delta \quad \text{ if } \delta<s\wedge t\end{cases}.\end{aligned}$$

This shows that $[xy;s\wedge t]\in_\gamma \vee q_\delta \mu$. Next we let $x,a,y\in G$ such that $[a;s]\in_\gamma \mu$ and by (2.2)

$$\mu((xa)y)\vee\gamma \geq \mu(a)\wedge\delta$$
$$\geq s\wedge\delta$$
$$=\begin{cases} s \text{ if } s\leq\delta \\ \delta \text{ if } \delta<s \end{cases}.$$

From this we can say that $[(xa)y;s]\in_\gamma \vee q_\delta \mu$. Consequently μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

4.5. Theorem

Let μ be a fuzzy subset of G. Then the following conditions are equivalent:

(1) μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

(2) $U(\mu;t)(\neq\phi)$ is an interior ideal of G for all $t\in(\gamma,\delta]$.

Proof. Suppose that μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G and let $x,a,y\in G$ be such that $a\in U(\mu;t)$ for some $t\in(\gamma,\delta]$. Then $\mu(a)\geq t$ and by hypothesis

$$\mu((xa)y)\vee\gamma\geq\mu(a)\wedge\delta)$$
$$\geq t\wedge\delta$$
$$=t \text{ (since } t\in(\gamma,\delta]).$$

Hence $\mu((xa)y)\geq t>\gamma$, this shows that $(xa)y\in U(\mu;t)$ and hence $U(\mu;t)(\neq\phi)$ is an interior of G for all $x,a,y\in G$ and $t\in(\gamma,\delta]$.

Conversely, assume that $U(\mu;t)(\neq\phi)$ is an interior of G for all $t\in(\gamma,\delta]$. If there exist $x,a,y\in G$ such that $\mu((xa)y)\vee\gamma<\mu(a)\wedge\delta)$. Then choose $t\in(\gamma,\delta]$ such that $\mu((xa)y)\vee\gamma<t\leq\mu(a)\wedge\delta)$. From here we have $a\in U(\mu;t)$ but $(xa)y\notin U(\mu;t)$, a contradiction. Hence $\mu((xa)y)\vee\gamma\geq\mu(a)\wedge\delta)$ for all $x,a,y\in G$ and $t\in(\gamma,\delta]$. The remaining proof is a consequence of the Theorem 3.9 . Therefore μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

4.6. Proposition

Every $(\in_\gamma,\in_\gamma)$-fuzzy interior ideal of G is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Proof Follows from Proposition 3.13.

4.7. Remark

A fuzzy subset μ of an AG-groupoid G is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G if and only if it satisfies conditions (2.1), and (2.2) of Theorem 4.4.

4.8. Proposition

Every $(\in_\gamma \vee q_\delta,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G.

Proof The proof is straightforward and is omitted here.

Note that, every $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy ideal of G is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal. Similarly, every $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of G. In the following propositions, we provide the conditions under which the converses of the above statements are true.

4.9. Proposition

If G is an intra-regular AG-groupoid, then every $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy ideal of G.

Proof Let μ be an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy interior ideal of G. Let $a,b\in G$. Since G is an intra-regular, there exist $x,y\in G$ such that $a=(xa^2)y$. Thus,

$$\mu(ab)\vee\gamma=\mu(((xa^2)y)b)\vee\gamma$$
$$=\mu((by)(xa^2))\vee\gamma$$
$$=\mu((by)(x(aa)))\vee\gamma$$
$$=\mu((by)(a(xa))\vee\gamma$$
$$=\mu((by)a(xa)))\vee\gamma$$
$$\geq\mu(a)\wedge\delta.$$

Similarly, we can show that $\mu(ab)\vee\gamma\geq\mu(b)\wedge\delta$ and hence, μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy ideal of G.

4.10. Proposition

If G is an intra-regular AG-groupoid, then every $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

Proof Let μ be an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy generalized bi-ideal of G. Let $a,b\in G$. Since G is intra-regular, there exist $x,y\in G$ such that $a=(xa^2)y$. Thus,

$$\mu(ab)\vee\gamma=\mu(((xa^2)y)b)\vee\gamma=\mu(((xa^2)(ey))b)\vee\gamma$$
$$=\mu(((ye)(a^2x)b)\vee\gamma=\mu((a^2((ye)x))b)\vee\gamma$$
$$=\mu(((aa)((ye)x))b)\vee\gamma=\mu(((x(ye))(aa))b)\vee\gamma$$
$$=\mu((a((x(ye)a))b)\vee\gamma\geq\mu(a)\wedge\mu(b)\wedge\delta.$$

Thus, μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy AG-subgroupoid of G and consequently, μ is an $(\in_\gamma,\in_\gamma \vee q_\delta)$-fuzzy bi-ideal of G.

In Propositions 4.9 and 4.10, if we take G as regular, then the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy interior ideals, $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy generalized bi-ideals and $(\in_\gamma, \in_\gamma \vee q_\delta)$ -fuzzy bi-ideals coincide.

5. $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy bi- (interior) ideals

In this section, we define $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy bi-ideals, $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy interior ideals of AG -groupoids and investigate some important properties of AG -groupoids based on these new types of fuzzy bi-ideals and fuzzy interior ideals. It is important to note that level subsets are used to link ordinary ideals and fuzzy ideals of type $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$. If μ is a fuzzy subset of G and $J = \{t \mid t \in (0,1]$ and $U(\mu;t)$ is level subset of μ, then μ will be fuzzy bi-ideal (resp. an interior ideals)? depends on level subset whether it is empty or bi-ideal (resp. interior ideal)? Further, if μ is a fuzzy subset and $J = \{t \mid t \in (0,1]$ and $U(\mu;t)$ is an empty set or a bi-ideal (resp. interior ideal) of $G\}$, then we consider the following questions:

(a) if $J = (\delta, 1]$, what type of fuzzy bi-ideals (fuzzy interior ideal) of G will μ be ?

(b) if $J = (\gamma, \delta]$, $(\gamma, \delta \in (0,1])$, will μ be a kind of fuzzy bi-ideal (resp. fuzzy interior ideal) of G or not?

In the following, we give the answers to these questions.

5.1. Definition .

A fuzzy subset μ of G is called an $(\overline{\beta}, \overline{\alpha})$ *-fuzzy bi-ideal* of G if it satisfies the following conditions:

$(B5)$ $(\forall x, y \in G)(\forall s, t \in (\gamma, 1])$ $([xy; s \wedge t]\overline{\beta}\mu \rightarrow [x;s]\overline{\alpha}\mu$ or [illegible]

$(B6)$ $(\forall x, a, y \in G)(\forall s, t \in (\gamma, 1])$ $([(xa)y; s \wedge t]\overline{\beta}\mu \rightarrow [x;s]\overline{\alpha}\mu$ or [illegible]

5.2. Theorem

If μ is a fuzzy subset of G, then the following conditions are equivalent:

(1) μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy bi-ideal of G.

(2) μ satisfies the following conditions:

(i) $(\forall x, y \in G)(\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y))$,

(ii) $(\forall x, a, z \in G)(\mu((xa)z) \vee \delta \geq \mu(x) \wedge \mu(z))$.

Proof. Let μ be an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$ -fuzzy bi-ideal of G. If there exist $x, y \in G$ such that $\mu(xy) \vee \delta < \mu(x) \wedge \mu(y)$, then $\mu(xy) \vee \delta < t \leq \mu(x) \wedge \mu(y)$ for some $t \in (\gamma, 1]$, we see that $[xy;t]\overline{\in}_\gamma \mu$ but $[x;t] \in_\gamma \mu$ and $[y;t] \in_\gamma \mu$, a contradiction and hence $\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$. Next we suppose that there exist $a, b, c \in G$ such that $\mu((ab)c) \vee \delta < \mu(a) \wedge \mu(c)$, then $\mu((ab)c) \vee \delta < s \leq \mu(a) \wedge \mu(c)$ for some $s \in (\gamma, 1]$, shows that $[(ab)c;t]\overline{\in}_\gamma \mu$ but $[a;t] \in_\gamma \mu$ and $[c;t] \in_\gamma \mu$, a contradiction, therefore $\mu((xa)z) \vee \delta \geq \mu(x) \wedge \mu(z)$ for all $x, a, z \in G$.

Conversely; suppose μ satisfies conditions (i) and (ii). On the contrary assume that for $x, y \in G$, $[xy; s \wedge t]\overline{\in}_\gamma \mu$ such that $[x;s]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$ and $[y;t]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu.$ Hence $\mu(xy) < s \wedge t, \mu(x) \geq s, \mu(x) + s > 2\delta, \mu(y) \geq t$ and $\mu(y) + t > 2\delta$. We claim that $\mu(x) > \delta$ and $\mu(y) > \delta$. This is because, if $\mu(x) < \delta$ and $\mu(y) < \delta$, then $s \leq \mu(x) < \delta$ implies that $s < \delta$, similarly $t < \delta$. Thus $\mu(x) + s \leq \delta + s \leq \delta + \delta = 2\delta$ which leads to $[x;s]\overline{q}_\delta \mu$. Hence $[x;s]\overline{\in}_\gamma \vee \overline{q}_\delta \mu$, a contradiction to our assumption. Therefore $\mu(x) > \delta$ and $\mu(y) > \delta$. Hence

$$\begin{aligned} \mu(x) \wedge \mu(y) &> (s \wedge t) \vee \delta \\ &> \mu(xy) \vee \delta. \end{aligned}$$

Thus, a contradiction to (i). This is due to our wrong supposition. Therefore for $x, y \in G$ such that $[xy; s \wedge t]\overline{\in}_\gamma \mu$ then it implies that $[x;s]\overline{\in}_\gamma \vee \overline{q}_\delta \mu$ or $[y;t]\overline{\in}_\gamma \vee \overline{q}_\delta \mu.$ Similarly, for $x, a, y \in G$, $[(xa)y; s \wedge t]\overline{\in}_\gamma \mu$ such that $[x;s]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu$ and $[y;t]\overline{\overline{\in}_\gamma \vee \overline{q}_\delta}\mu.$ Therefore $\mu((xa)y) < s \wedge t, \mu(x) \geq s, \mu(x) + s > 2\delta, \mu(y) \geq t$ and $\mu(y) + t > 2\delta$. We claim that $\mu(x) > \delta$ and $\mu(y) > \delta$. This is because, if $\mu(x) < \delta$ and $\mu(y) < \delta$, then $s \leq \mu(x) < \delta$ implies that $s < \delta$, similarly $t < \delta$. Hence $\mu(x) + s \leq \delta + s \leq \delta + \delta = 2\delta$ implies that $[x;s]\overline{q}_\delta \mu.$ Hence $[x;s]\overline{\in}_\gamma \vee \overline{q}_\delta \mu$, a contradiction. Therefore $\mu(x) > \delta$ and $\mu(y) > \delta$. Hence

$$\mu(x) \wedge \mu(y) > (s \wedge t) \vee \delta$$
$$> \mu((xa)y) \vee \delta.$$

Thus, a contradiction to (ii). Therefore for $x, y \in G$ such that $[(xa)y; s \wedge t]\overline{\in}_\gamma \mu$ then it implies that $[x;s]\overline{\in}_\gamma \vee \overline{q}_\delta \mu$ or $[y;t]\overline{\in}_\gamma \vee \overline{q}_\delta \mu$. Hence, μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal of G.

5.3. Definition

A fuzzy subset μ of G is called an $(\overline{\beta}, \overline{\alpha})$*-fuzzy interior ideal* of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(\forall s, t \in (\gamma, 1])$ $([xy; s \wedge t]\overline{\beta}\mu \rightarrow [x;s]\overline{\alpha}\mu$ or $[y;t]\overline{\alpha}\mu)$,

(2) $(\forall x, a, y \in G)(\forall s \in (\gamma, 1])$ $([(xa)y; s]\overline{\beta}\mu \rightarrow [a;s]\overline{\alpha}\mu)$.

5.4. Theorem

If μ is a fuzzy subset of G, then the following conditions are equivalent:

(1) μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideal of G.

(2) μ satisfies the following conditions:

(i) $(\forall x, y \in G)(\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y))$,

(ii) $(\forall x, a, z \in G)(\mu((xa)z) \vee \delta \geq \mu(a))$.

Proof It is an immediate consequence of Theorem 5.2.

5.5. Lemma

If μ is a fuzzy subset of G, then the following are equivalent:

(1) $U(\mu;t)$ is a bi-ideal of G for all $t \in (\delta, 1]$.

(2) μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi-ideal.

Proof. Assume that $U(\mu;t)$ is a bi-ideal of G for all $t \in (\delta, 1]$. If there exist $x, y \in G$ such that $\mu(xy) \vee \delta < \mu(x) \wedge \mu(y) = t_1$, then $t_1 \in (\delta, 1]$, $x, y \in U(\mu; t_1)$. But $\mu(xy) < t_1$ implies $xy \notin U(\mu; t_1)$, a contradiction. Hence $\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

If there exist $x, y, z \in G$ such that $\mu((xy)z) \vee \delta < \mu(x) \wedge \mu(z) = t_2$, then $t_2 \in (\delta, 1]$, $x, z \in U(\mu; t_2)$. But $\mu((xy)z) < t_2$ implies $(xy)z \notin U(\mu; t_2)$, a contradiction. Hence $\mu((xy)z) \vee \delta \geq \mu(x) \wedge \mu(z)$ for all $x, y, z \in G$.

Conversely, suppose that for $x, y \in U(\mu;t)$, by Theorem 5.2 we get

$$\mu(xy) \vee \delta \geq \mu(x) \wedge \mu(y) \geq t,$$

and so $\mu(xy) \geq t$. It follows that $xy \in U(\mu;t)$. Let $x, z \in U(\mu;t)$, then $\mu(x) \geq t$ and $\mu(z) \geq t$. By Theorem 5.2, we get

$$\mu((xy)z) \vee \delta \geq \mu(x) \wedge \mu(z) \geq t,$$

and so $\mu((xy)z) \geq t$. It follows that $(xy)z \in U(\mu;t)$. Thus $U(\mu;t)$ is a bi-ideal of G for all $t \in (\delta, 1]$.

5.6. Lemma

If μ is a fuzzy subset of G, then the following are equivalent:

(1) $U(\mu;t)$ is an interior ideal of G for all $t \in (\delta, 1]$.

(2) μ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy interior ideal.

Proof This follows directly from Lemma 5.5.

Next we discuss fuzzy bi-ideals with thresholds and fuzzy interior ideals with thresholds.

5.7. Definition

Let $\gamma, \delta \in (0,1]$ and $\gamma < \delta$, then a fuzzy subset μ of G is called a *fuzzy bi-ideal with thresholds* $(\gamma, \delta]$ of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,

(2) $(\forall x, y, z \in G)$ $(\mu((xy)z) \vee \gamma \geq \mu(x) \wedge \mu(z) \wedge \delta)$.

5.8. Theorem

A fuzzy subset μ of G is a fuzzy bi-ideal with thresholds $(\gamma, \delta]$ of G if and only if $U(\mu;t)(\neq \phi)$ is a bi-ideal of G for all $\gamma < t \leq \delta$.

5.9. Definition

Let $\gamma, \delta \in (0,1]$ and $\gamma < \delta$, then a fuzy subset μ of G is called a *fuzzy interior ideal with thresholds* $(\gamma, \delta]$ of G if it satisfies the following conditions:

(1) $(\forall x, y \in G)(\mu(xy) \vee \gamma \geq \mu(x) \wedge \mu(y) \wedge \delta)$,

(2) $(\forall x, y, z \in G)(\mu((xy)z) \vee \gamma \geq \mu(a) \wedge \delta)$.

5.10. Theorem

A fuzzy subset μ of G is a fuzzy interior ideal with thresholds $(\gamma, \delta]$ of G if and only if $U(\mu;t)(\neq \phi)$ is an interior ideal of G for all $\gamma < t \leq \delta$.

6. Conclusion

In science and technology, the use of algebraic structures play an unavoidable role. For instance, semigroups are basic structures in computer science, control engineering etc. Ordered semigroups are used in fuzzy automata, formal languages, coding theory etc. AG-groupoids are used in Flocks (Physics, Biology) theory. Due to these diverse applications, algebraic structures especially AG-groupoids and related structures are presently a central focus for the researchers. In this paper, we studied generalized fuzzy bi-ideals (resp. ideals and interior ideals) and provided different characterization theorems of AG-groupoids in terms $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy bi- (interior) ideals and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy bi- (interior) ideals. Characteristic function and level subset of μ are used to show the connection between ordinary bi-ideals (resp. interior ideals) and fuzzy bi-ideals (resp. fuzzy interior interior deals) of type $(\in_\gamma, \in_\gamma \vee q_\delta)$. In particular, if $J = \{t \mid t \in (0,1]$ and $U(\mu;t)$ is an empty set or a bi-ideal (resp. an ideal or an interior ideal) of $G\}$, we discussed what kind of fuzzy bi-ideals (resp. fuzzy ideals or fuzzy interior ideals) of G will μ be.

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