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Introduction of σ-Statistical Convergence and Lacunary σ-Statistical Convergence

*Dr. Rajeev Kumar and **Preety

*Associate Professor, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
**Research Scholar, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
Email: preetyyadav0066@gmail.com

Abstract: The main object of this chapter is to study two more extensions of the concept of statistical convergence namely σ-statistical convergence and lacunary σ-statistical convergence. We also study the concept of L_θ-convergence. In section 1.2 we study some inclusion relations between L_θ-convergence and lacunary σ-statistical convergence and show that these are equivalent for bounded sequences. Further in section 1.3 we study relation between σ-statistical convergence and lacunary σ-statistical convergence.

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Introduction:

Convergence in distribution (sometimes called convergence in law) is based on the distribution of random variables, rather than the individual variables themselves. It is the convergence of a sequence of cumulative distribution functions (CDF). As it's the CDFs, and not the individual variables that converge, the variables can have different probability spaces.

In more formal terms, a sequence of random variables converges in distribution if the CDFs for that sequence converge into a single CDF. Let's say you had a series of random variables, Xn. Each of these variables X1, X2,…Xn has a CDF FXn (x), which gives us a series of CDFs {FXn (x)}. Convergence in distribution implies that the CDFs converge to a single CDF, Fx (x) (Kapadia et. al, 2017).

Several methods are available for proving convergence in distribution. For example, Slutsky's Theorem and the Delta Method can both help to establish convergence. Convergence of moment generating functions can prove convergence in distribution, but the converse isn't true: lack of converging MGFs does not indicate lack of convergence in distribution. Scheffe's Theorem is another alternative, which is stated as follows (Knight, 1999, p.126).

In undergraduate courses we often teach the following version of the central limit theorem: if X1,..., Xn are an iid sample from a population with mean μ and standard deviation σ then n 1/2 ($\bar{X}$ − μ)/σ has approximately a standard normal distribution. Also we say that a Binomial (n, p) random variable has approximately a N (np, np (1 − p)) distribution. What is the precise meaning of statements like "X and Y have approximately the same distribution"? The desired meaning is that X and Y have nearly the same cdf. But care is needed. Here are some questions designed to try to highlight why care is needed.

Definition 1.1.1. Let σ be a mapping of the set of positive integers into itself. A continuous linear functional Φ on l_∞, the space of real bounded sequences x = $\{\xi_k\}$, is said to be an invariant mean or a σ-mean if and only if

1. $\Phi(x) \geq 0$ if $\xi_k \geq 0$ for all k,
2. $\Phi(\{\xi_{\sigma(k)}\}) = \Phi(x)$ for all $x \in l_\infty$,
3. Φ(e) = 1 where e = {1,1,1,…}.

The mappings σ are one-to-one and such that $\sigma^m(k) \neq k$ for all positive integers k and m, where $\sigma^m(k)$ denotes the m^{th} iterate of the mapping σ at k. Thus Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x) = \lim \xi_k$ for all $x \in c$. In case σ is the translation mapping k→k+1, an invariant mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [19].

If $x = \{\xi_k\}$, set $Tx = \{T\xi_k\} = \{\xi_{\sigma(k)}\}$. It can be shown [28] that

$$V_\sigma = \{x = \{\xi_k\}: \lim_{m\to\infty} t_{mk}(x) = \xi e \text{ uniformly in k}, \xi = \sigma\text{-lim } \xi_k\}$$

where $t_{mk}(x) = \dfrac{(\xi_k + T\xi_k + \ldots + T^m\xi_k)}{m+1}$.

Several authors including Mursaleen [22], Savas [27], Schaefer [31] and others have studied invariant convergent sequences.

Definition 1.1.2. A sequence $x = \{\xi_k\}$ is said to be strongly σ-convergent [23] to ξ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.}$$

In this case we write $\xi_k \to \xi[V_\sigma]$ and $[V_\sigma]$ denotes the set of all strongly σ-convergent sequences.

Remark 1.1.3.

(i) For $\sigma(m) = m+1$, the space $[V_\sigma]$ is the space of strongly almost convergent sequences.

(ii) It is known [23] that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

Definition 1.1.4. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this chapter the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 1.1.5. Let θ be a lacunary sequence. The space denoted by N_θ is defined [9] as

$$N_\theta = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_k - \xi| = 0\}.$$

Definition 1.1.1. A sequence $x = \{\xi_k\}$ is said to be lacunary strong σ-convergent [28] to ξ if

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.}$$

We shall denote by L_θ the set of all lacunary strong σ-convergent sequences.

Remark 1.1.1. $L_\theta \Leftrightarrow [V_\sigma]$ for every lacunary sequence θ.

Definition 1.1.1. A complex number sequence $x = \{\xi_k\}$ is said to be σ-statistically convergent or S_σ-convergent to the number ξ if for each ε > 0

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \le k \le n: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0$$

uniformly in m.

In this case we write $S_\sigma\text{-}\lim \xi_k = \xi$ or $\xi_k \to \xi(S_\sigma)$ and S_σ denotes the set of all σ-statistically convergent sequences.

Definition 1.1.9. Let $\theta = \{k_r\}$ be a lacunary sequence. The complex number sequence $x = \{\xi_k\}$ is said to be lacunary σ-statistically convergent or $S_{\sigma\theta}$-convergent to the number ξ if for each ε > 0

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0$$

uniformly in m.

In this case we write $S_{\sigma\theta}\text{-}\lim \xi_k = \xi$ or $\xi_k \to \xi(S_{\sigma\theta})$ and $S_{\sigma\theta}$ denotes the set of all lacunary σ-statistically convergent sequences.

1.2 Some Inclusion Relations Between L_θ-Convergence And Lacunary σ-Statistical Convergence

In his section we study some inclusion relations between L_θ-convergence and lacunary σ-statistical convergence and show that these are equivalent for bounded sequences.

Theorem 1.4.1. Let $\theta = \{k_r\}$ be a lacunary sequence. Then

(i) $\xi_k \to \xi(L_\theta) \Rightarrow \xi_k \to \xi(S_{\sigma\theta})$,

(ii) if $x \in l_\infty$ and $\xi_k \to \xi(S_{\sigma\theta})$, then $\xi_k \to \xi(L_\theta)$,

(iii) $S_{\sigma\theta} \cap l_\infty = L_\theta$.

Proof. (i). Since $\xi_k \to \xi(L_\theta)$, for each ε > 0, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.} \qquad \ldots(1)$$

If ε > 0, we can write

$$\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| \ge \sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon}} |\xi_{\sigma^k(m)} - \xi|$$

$$\ge \varepsilon |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

Consequently,

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

Hence by (1) and the fact that ε is fixed number, we have

$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| = 0$ uniformly in m,
i.e. $\xi_k \to \xi(S_{\sigma\theta})$.

(ii). Suppose that $\xi_k \to \xi(S_{\sigma\theta})$ and $x \in l_\infty$. Then for each $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| = 0 \text{ uniformly in m.} \quad \text{... (2)}$$

Since $x \in l_\infty$, there exists a positive real number M such that $|\xi_{\sigma^k(m)} - \xi| \leq M$ for all k and m.
For given $\varepsilon > 0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r}|\xi_{\sigma^k(m)} - \xi| = \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)}-\xi|\geq\varepsilon}}|\xi_{\sigma^k(m)} - \xi| + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)}-\xi|<\varepsilon}}|\xi_{\sigma^k(m)} - \xi|$$

$$\leq \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)}-\xi|\geq\varepsilon}} M + \frac{1}{h_r}\sum_{k\in I_r}\varepsilon = \frac{M}{h_r}|\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon\frac{1}{h_r}[n-(n-h_r+1)+1]$$

$$= \frac{M}{h_r}|\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon\frac{1}{h_r}h_r$$

$$= \frac{M}{h_r}|\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon$$

$$\Rightarrow \lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\xi_{\sigma^k(m)} - \xi| \leq M\lim_{r\to\infty}\frac{1}{h_r}|\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon$$

Hence by using (2), we get

$$\lim_{r\to\infty}\frac{1}{h_r}\sum_{k\in I_r}|\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.} \quad \text{...(3)}$$

$\Rightarrow$ $\xi_k \to \xi(L_\theta)$.

Example 1.2.2. Let θ be given and define ξ_k to be $1,2,3,\ldots,[\sqrt{h_r}]$ for $k = \sigma^n(m)$, $n = k_{r-1} + 1, k_{r-1} + 2, \ldots, k_{r-1} + [\sqrt{h_r}]$; $m \geq 1$ and $\xi_k = 0$ otherwise (where [ ] denotes the greatest integer function).
Note that x is not bounded. Now

$$\frac{1}{h_r}|\{k \in I_r: |\xi_{\sigma^k(m)} - 0| \geq \varepsilon\}| = \frac{[\sqrt{h_r}]}{h_r} \to 0 \text{ as } r \to \infty,$$

i.e. $\xi_k \to 0(S_{\sigma\theta})$. But

$$\frac{1}{h_r}\sum_{k\in I_r}|\xi_{\sigma^k(m)} - 0| = \frac{1}{h_r}\left([\sqrt{h_r}]\frac{([\sqrt{h_r}]+1)}{2}\right) \to \frac{1}{2} \neq 0 \text{ as } r \to \infty,$$

i.e. ξ_k □$0(L_\theta)$.

Thus inclusion in (i) is proper and this example shows that the boundedness condition can not be omitted from (ii).

(iii). It follows from (i), (ii), Remark 1.1.7 and the fact that $[V_\sigma] \subset l_\infty$.

This completes the proof of the theorem.

1.3 In this section we study relation between S_σ-convergence and $S_{\sigma\theta}$-convergence. First we discuss a lemma which will be used in studying that relation.

Lemma 1.4.1. A sequence $x = \{\xi_k\}$ is σ-statistically convergent to the number ξ if for given $\varepsilon_1 > 0$ and each $\varepsilon > 0$, there exist n_0 and m_0 such that

$$\frac{1}{n}|\{0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| < \varepsilon_1$$

for all $n \ge n_0$ and $m \ge m_0$.

Proof. Let $\varepsilon_1 > 0$ be given. For each $\varepsilon > 0$, choose n_0' and m_0 such that

$$\frac{1}{n}|\{0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| < \frac{\varepsilon_1}{2} \qquad \ldots(4)$$

for all $n \ge n_0'$ and $m \ge m_0$.

It is enough to prove that there exists n_0'' such that for $n \ge n_0''$, $0 \le m \le m_0$,

$$\frac{1}{n}|\{0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| < \varepsilon_1 \qquad \ldots(5)$$

since taking $n_0 = \max\{n_0', n_0''\}$, (5) will hold for $n \ge n_0$ and for all m, which gives the result.

Once m_0 has been chosen, $0 \le m \le m_0$, m_0 is fixed.

So let $|\{0 \le k \le m_0-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = K$.

Now taking $0 \le m \le m_0$ and $n \ge m_0$, we have

$$\frac{1}{n}|\{0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = \frac{1}{n}|\{0 \le k \le m_0-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

$$+ \frac{1}{n}|\{m_0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

$$\le \frac{1}{n}K + \frac{\varepsilon_1}{2} \qquad \text{[Using (4)]}$$

$$< \varepsilon_1 \qquad \text{[Taking n sufficiently large]}$$

which gives (5), and hence the result follows.

Theorem 1.3.2. $S_{\sigma\theta} = S_\sigma$ for every lacunary sequence θ.

Proof. Let $x \in S_{\sigma\theta}$. Then from Definition 1.1.9, given $\varepsilon_1 > 0$, there exist r_0 and ξ such that

$$\frac{1}{h_r}|\{0 \le k \le h_r-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| < \varepsilon_1$$

for $r \ge r_0$ and $m = k_{r-1} + 1 + u$, $u \ge 0$.

Let $n \ge h_r$ and write $n = ih_r + t$ where $0 \le t \le h_r$, i is an integer. Since $n \ge h_r$, it follows that $i \ge 1$.

Now

$$\frac{1}{n}|\{0 \le k \le n-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| \le \frac{1}{n}|\{0 \le k \le (i+1)h_r-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

$$= \frac{1}{n}\sum_{j=0}^{i}|\{jh_r \le k \le (j+1)h_r-1 : |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

$$\le \frac{1}{n}(i+1)h_r\varepsilon_1$$

$$\le 2ih_r\frac{\varepsilon_1}{n} \qquad [i \ge 1]$$

for $\frac{h_r}{n} \le 1$, since $\frac{ih_r}{n} \le 1$. So

$\frac{1}{n}|\{0 \le k \le n-1: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| \le 2\varepsilon_1$.

Then, by Lemma 1.4.1, $x \in S_\sigma$.

Thus $S_{\sigma\theta} \subset S_\sigma$.

It is easy to see that $S_\sigma \subset S_{\sigma\theta}$.

Hence $S_{\sigma\theta} = S_\sigma$ for every lacunary sequence θ.

This completes the proof of the theorem.

Remark 1.3.3. When $\sigma(m) = m + 1$, from Definition 1.1.8 and Definition 1.1.9, we have the definitions of almost statistical convergence and lacunary almost statistical convergence of a sequence.

Corresponding author:
Mrs. Preety
Research Scholar, Department of Mathematics,
OPJS University, Churu,
Rajasthan (India)
Contact No. +91-9992845999
Email- preetyyadav0066@gmail.com

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