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marslandresearcher@gmail.com

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MARSLAND PRESS
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Characterization Of Statistical R-Convergence

*Dr. Rajeev Kumar and **Preety

*Associate Professor, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
**Research Scholar, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
Email: preetyyadav0066@gmail.com

Abstract: Convergence of random variables (sometimes called stochastic convergence) is where a set of numbers settle on a particular number. It works the same way as convergence anywhere else; For example, cars on a 5-line highway might converge to one specific lane if there's an accident closing down four of the other lanes. In the same way, a sequence of numbers (which could represent cars or anything else) can converge (mathematically, this time) on a single, specific number. Certain processes, distributions and events can result in convergence— which basically mean the values will get closer and closer together. When Random variables converge on a single number, they may not settle exactly that number, but they come very, very close. In notation, x (xn → x) tells us that a sequence of random variables (xn) converges to the value x.

[Kumar, R. and Preety. **Characterization Of Statistical R-Convergence.** *Researcher* 2020;12(7):39-42]. ISSN 1553-9865 (print); ISSN 2163-8950 (online). http://www.sciencepub.net/researcher. 8. doi:10.7537/marsrsj120720.08.

Keywords: Characterization, Statistical, Convergence

Introduction

Statistics is concerned with the collection and analysis of data and with making estimations and predictions from the data. Typically two branches of statistics are discerned: descriptive and inferential. Inferential statistics is usually used for two tasks: to estimate properties of a population given sample characteristics and to predict properties of a system given its past and current properties. To do this, specific statistical constructions were invented. The most popular and useful of them are the average or mean (or more exactly, arithmetic mean) m and standard deviation s (variance s 2). To make predictions for future, statistics accumulates data for some period of time. To know about the whole population, samples are used. Normally such inferences (for future or for population) are based on some assumptions on limit processes and their convergence. Iterative processes are used widely in statistics. For instance the empirical approach to probability is based on the law (or better to say, conjecture) of big numbers, states that a procedure repeated again and again, the relative frequency probability tends to approach the actual probability. The foundation for estimating population parameters and hypothesis testing is formed by the central limit theorem, which tells us how sample means change when the sample size grows. In experiments, scientists measure how statistical characteristics (e.g., means or standard deviations) converge (cf., for example, [23, 31]). Convergence of means/averages and standard deviations have been studied by many authors and applied to different problems (cf. [1-4, 17, 19, 20, 24-28, 35]). Convergence of statistical characteristics such as the average/mean and standard deviation are related to statistical convergence as we show in this section.

Let m and c be the spaces of all bounded and convergent real sequences x = (xk) normed by x = supn |xn|, respectively. Let B be the class of (necessarily continuous) linear functionals β on m which are nonnegative and regular, that is, if x ≥ 0, (i.e., xk ≥ 0 for all k ∈ N:= {1, 2,...}) then β(x) ≥ 0, and β(x) = limk xk, for each x ∈ c. If β has the additional property that β(σ(x)) = β(x) for all x ∈ m, where σ is the left shift operator, defined by σ(x1, x2,...)=(x2, x3,...) then β is called a Banach limit. The existence of Banach limits has been shown by Banach [2,17,19], and another proof may be found in [3]. It is well known [21] that the space of all almost convergent sequences can be represented as the set of all x ∈ m which have the same value under any Banach limit. In the research, we study some generalized limits so that the space of all bounded statistically convergent sequences can be represented as the set of all bounded sequences which have the same value under any such limit. It is proved that the set of such limits and the set of Banach limits are distinct but their intersection is not empty.

In this section we study a useful characterization of statistical r-convergence and some more results.

Theorem 5.5.1. A sequence $x = \{\xi_k\}$ is statistically r-convergent if and only if every statistically dense subsequence of it is statistically r-convergent.

Proof. First suppose that st-r-lim $\xi_k = \xi$. Let us take a statistically dense subsequence $y = \{\xi_{k_n}\}$ of x and assume that it is statistically r-divergent.

Then for any real number ξ, there is some $\varepsilon > 0$ such that

$\delta(B_{r,\varepsilon}) > 0$

where $B_{r,\varepsilon} = \{k_n \in \mathbb{N}: |\xi_{k_n} - \xi| > r + \varepsilon\}$.

As y is a subsequence of x, we have

$A_{r,\varepsilon} \supseteq B_{r,\varepsilon}$

where $A_{r,\varepsilon} = \{k \in \mathbb{N}: |\xi_k - \xi| > r + \varepsilon\}$.

Consequently, $\delta(A_{r,\varepsilon}) \geq \delta(B_{r,\varepsilon}) > 0$

as the subsequence y is statistically dense in x.

This contradicts the fact that x is statistically r-convergent.

Hence y is also statistically r-convergent.

Conversely, suppose that every statistically dense subsequence of x is statistically r-convergent. Then x is also statistically r-convergent since x is a statistically dense subsequence of itself.

This completes the proof of the theorem.

Corollary 5.2.2. A statistically r-convergent sequence contains not only dense statistically r-convergent subsequences, but also dense r-convergent subsequences.

Theorem 5.2.5. A sequence $x = \{\xi_k\}$ is statistically r-convergent ξ to if and only if there exists a set $K = \{k_1 < k_2 <\ldots< k_n <\ldots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and r-lim $\xi_{k_n} = \xi$.

Proof. First suppose that st-r-lim $\xi_k = \xi$.

Consider the sets $K_{r,j} = \{k \in \mathbb{N}: |\xi_k - \xi| < r + \frac{1}{j}\}$ for all $j = 1,2,3\ldots$.

As $K_{r,j} = \mathbb{N} - \{k \in \mathbb{N}: |\xi_k - \xi| \geq r + \frac{1}{j}\}$ and x is statistically r-convergent to ξ, we have

$\delta(K_{r,j}) = 1 \quad j = 1,2,3\ldots$ ... (1)

Now

$K_{r,j+1} = \{k \in \mathbb{N}: |\xi_k - \xi| < r + \frac{1}{j+1}\}$

$\subset \{k \in \mathbb{N}: |\xi_k - \xi| < r + \frac{1}{j}\}$

$= K_{r,j}$.

So

$K_{r,j+1} \subset K_{r,j}$ for all $j = 1,2,3\ldots$ …(2)

Let us choose an arbitrary number $v_1 \in K_{r,1}$. Then according to (1) and (2), $\exists\, v_2 > v_1$, $v_2 \in K_{r,2}$ such that

$\frac{1}{n}|\{k \leq n: |\xi_k - \xi| < r + \frac{1}{2}\}| > \frac{1}{2}$ for all $n \geq v_2$.

In a similar way, $\exists\, v_3 > v_2$, $v_3 \in K_{r,3}$ such that

$\frac{1}{n}|\{k \leq n: |\xi_k - \xi| < r + \frac{1}{3}\}| > \frac{2}{3}$ for all $n \geq v_5$.

We continue this process and construct by induction a sequence

$v_1 < v_2 <\ldots< v_j <\ldots$

of positive integers such that for $j = 1,2,3,\ldots$

$v_j \in K_{r,j}$ and

$\frac{1}{n}|\{k \leq n: |\xi_k - \xi| < r + \frac{1}{j}\}| > \frac{j-1}{j}$ for all $n \geq v_j$. …(3)

Now we construct the set K as follows:

$K = \{k \in \mathbb{N}: 1 \leq k \leq v_1\} \cup (\bigcup_{j\in\mathbb{N}} \{k \in K_{r,j}: v_j \leq k \leq v_{j+1}\})\ldots$ (4)

Then from (2), (3) and (4) we conclude that for all n from the interval $v_j \leq n \leq v_{j+1}$ and for all $j = 1,2,3,\ldots$, we have

$\frac{1}{n}|\{k \leq n: k \in K\}| = \frac{1}{n}|\{k \leq n: |\xi_k - \xi| < r + \frac{1}{j}\}| > \frac{j-1}{j}$.

Hence it follows that $\delta(K) = 1$. Take some $\varepsilon > 0$ and choose a number $j \in \mathbb{N}$ such that $\frac{1}{j} < \varepsilon$. If $n \in K$ and $n \geq v_j$, then, by definition of K, there exists a number $m \geq j$ such that $v_m \leq n \leq v_{m+1}$ and thus $n \in K_{r,m}$. Hence we have

$|\xi_n - \xi| < r + \frac{1}{j} < r + \varepsilon$.

As this is true for all $n \in K$, we see that r-$\lim_{\substack{k\to\infty \\ k\in K}} \xi_k = \xi$.

Conversely, suppose that there exists a set $K = \{k_1 < k_2 <\ldots< k_n <\ldots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and r-lim $\xi_{k_n} = \xi$. Then for given $\varepsilon > 0$ there is a number n such that for each $k \in K$

$|\xi_k - \xi| < r + \varepsilon \ \forall\, k \geq n$. …(5)

Put $A_{r,\varepsilon} = \{k \in \mathbb{N}: |\xi_k - \xi| \geq r + \varepsilon\}$.

Then we have

$A_{r,\varepsilon} \subseteq \mathbb{N} - \{k_n, k_{n+1}, k_{n+2}, \ldots\}$.

Since $\delta(K) = 1$, we get $\delta(\mathbb{N} - \{k_n, k_{n+1}, k_{n+2}, \ldots\}) = 0$.

Thus $\delta(A_{r,\varepsilon}) = 0$ for each $\varepsilon > 0$.

$\Rightarrow$ st-r-lim $\xi_k = \xi$.

This completes the proof of the theorem.

Corollary 5.2.5. A sequence $x = \{\xi_k\}$ is statistically r-convergent to ξ if and only if there exists a sequence $y = \{\eta_k\}$ such that $\delta(\{k \in \mathbb{N}: \eta_k = \xi_k\}) = 1$ and r-lim $\eta_k = \xi$.

Corollary 5.2.5. The following statements are equivalent:

1. st-r-lim $\xi_k = \xi$;
2. There is a set $K = \{k_1 < k_2 < \ldots < k_n < \ldots\} \subseteq \mathbb{N}$ such that $\delta(K) = 1$ and r-lim $\xi_k = \xi$;
3. For each $\varepsilon > 0$, there exists a set $K \subseteq \mathbb{N}$ and a number $m \in K$ such that $\delta(K) = 1$ and $|\xi_k - \xi| < r + \varepsilon$ for all $k \in K$ and $k \geq m$.

Notation. We denote the set of all statistical r-limits of a sequence $x = \{\xi_k\}$ by $L_{r\text{-st}}(x)$, i.e.

$L_{r\text{-st}}(x) = \{\xi \in \mathbf{R}: \text{st-r-lim } \xi_k = \xi\}$

Theorem 5.2.6. For every sequence $x = \{\xi_k\}$ and number $r \geq 0$, $L_{r\text{-st}}(x)$ is a convex subset of real numbers.

Proof. Let $\beta,\eta \in L_{r\text{-st}}(x)$ such that $\beta < \eta$ and $\xi \in [\beta,\eta]$. Then it is enough to prove that $\xi \in L_{r\text{-st}}(x)$.

Since $\xi \in [\beta,\eta]$, there is a number $\lambda \in [0,1]$ such that $\xi = \lambda\beta + (1-\lambda)\eta$.

As $\beta,\eta \in L_{r\text{-st}}(x)$, for each $\varepsilon > 0$ there exist index sets K_1, K_2 with $\delta(K_1) = \delta(K_2) = 1$ and positive integers n_1,n_2 such that

$|\xi_k - \beta| < r + \varepsilon$ for all $k \in K_1$ and $k \geq n_1$

$|\xi_k - \eta| < r + \varepsilon$ for all $k \in K_2$ and $k \geq n_2$.

Let us put $K = K_1 \cap K_2$ and $n = \max\{n_1,n_2\}$. Then, since intersection of two statistically dense sets is a statistically dense set, we have $\delta(K) = 1$.

Now for all $k \geq n$ with $k \in K$, we get

$$\begin{aligned}|\xi_k - \xi| &= |\xi_k - \lambda\beta - (1-\lambda)\eta| \\ &= |\xi_k + \lambda\xi_k - \lambda\xi_k - \lambda\beta - (1-\lambda)\eta| \\ &= |(\lambda\xi_k - \lambda\beta) + \{(1-\lambda)\xi_k - (1-\lambda)\eta\}| \\ &= |\lambda(\xi_k - \beta) + (1-\lambda)(\xi_k - \eta)| \\ &\leq \lambda|\xi_k - \beta| + (1-\lambda)|\xi_k - \eta| \\ &< \lambda(r + \varepsilon) + (1-\lambda)(r + \varepsilon) \\ &= r + \varepsilon\end{aligned}$$

So we conclude from Theorem 5.2.3 that st-r-lim $\xi_k = \xi$.

$\Rightarrow \xi \in L_{r\text{-st}}(x)$.

Hence $L_{r\text{-st}}(x)$ is a convex subset of real numbers.

This completes the proof of the theorem.

Lemma 5.2.7. If $q > r$, then $L_{r\text{-st}}(x) \subseteq L_{q\text{-st}}(x)$.

Proof. Let $\xi \in L_{r\text{-st}}(x)$. Then st-r-lim $\xi_k = \xi$.

Now by Lemma 5.1.4, st-q-lim $\xi_k = \xi$,

i.e. $\xi \in L_{q\text{-st}}(x)$.

Hence $L_{r\text{-st}}(x) \subseteq L_{q\text{-st}}(x)$.

This completes the proof of the lemma.

Let $x = \{\xi_k\}$ and $y = \{\eta_k\}$ be two sequences. Then their sum $x + y$ is equal to the sequence $\{\xi_k + \eta_k\}$ and their difference $x - y$ is equal to the sequence $\{\xi_k - \eta_k\}$.

Theorem 5.2.8. Let st-r-lim $\xi_k = \xi$ and st-q-lim $\eta_k = \eta$. Then

1. st-(r + q)-lim $\{\xi_k + \eta_k\} = \xi + \eta$;
2. st-(r + q)-lim $\{\xi_k - \eta_k\} = \xi - \eta$;
3. st-(|c| r)-lim $c\xi_k = c\xi$ for any $c \in \mathbf{R}$ where $cx = \{c\xi_k\}$.

Proof. 1. Since st-r-lim $\xi_k = \xi$, for every $\varepsilon > 0$ there exists a set $K_1 \subseteq \mathbb{N}$ and a number $m_1 \in K_1$ such that $\delta(K_1) = 1$ and

$$|\xi_k - \xi| < r + \frac{\varepsilon}{2} \quad \forall k \in K_1 \text{ and } k \geq m_1.$$

Also st-q-lim $\eta_k = \eta$, then for every $\varepsilon > 0$ there exists a set $K_2 \subseteq \mathbb{N}$ and a number $m_2 \in K_2$ such that $\delta(K_2) = 1$ and

$$|\eta_k - \eta| < q + \frac{\varepsilon}{2} \quad \forall k \in K_2 \text{ and } k \geq m_2.$$

Let $m = \max\{m_1, m_2\}$ and $K = K_1 \cup K_2$. Then $\delta(K) = 1$ and $\forall k \in K$ and $k \geq m$, we have

$$\begin{aligned}|(\xi_k + \eta_k) - (\xi + \eta)| &= |(\xi_k - \xi) + (\eta_k - \eta)| \\ &\leq |\xi_k - \xi| + |\eta_k - \eta| \\ &\leq r + \frac{\varepsilon}{2} + q + \frac{\varepsilon}{2} \\ &= r + q + \varepsilon.\end{aligned}$$

So by Theorem 5.2.3, we have

st-(r + q)-lim $\{\xi_k + \eta_k\} = \xi + \eta$.

2. From part (1), $\forall k \in K$ and $k \geq m$, we have

$$\begin{aligned}|(\xi_k - \eta_k) - (\xi - \eta)| &= |(\xi_k - \xi) - (\eta_k - \eta)| \\ &\leq |\xi_k - \xi| + |\eta_k - \eta| \\ &\leq r + \frac{\varepsilon}{2} + q + \frac{\varepsilon}{2} \\ &= r + q + \varepsilon.\end{aligned}$$

So by Theorem 5.2.3, we have

st-(r + q)-lim $\{\xi_k - \eta_k\} = \xi - \eta$.

5. Since st-r-lim $\xi_k = \xi$, for every $\varepsilon > 0$ there exists a set $K \subseteq \mathbb{N}$ and a number $m \in K$ such that $\delta(K) = 1$ and

$$|\xi_k - \xi| < r + \frac{\varepsilon}{|c|} \quad \forall k \in K \text{ and } k \geq m.$$

Now

$$\begin{aligned}|c\xi_k - c\xi| &= |c||\xi_k - \xi| \\ &< |c|\left(r + \frac{\varepsilon}{|c|}\right) \\ &= |c|r + \varepsilon.\end{aligned}$$

So by Theorem 5.2.3, we have
st-(|c| r)-lim $c\xi_k = c\xi$.

Corollary 5.2.9. If st-lim $\xi_k = \xi$ and st-lim $\eta_k = \eta$. Then

1. st-lim $\{\xi_k + \eta_k\} = \xi + \eta$;
2. st-lim $\{\xi_k - \eta_k\} = \xi - \eta$;
3. st-lim $c\xi_k = c\xi$ for any $c \in \mathbf{R}$.

Corresponding author:
Mrs. Preety
Research Scholar, Department of Mathematics,
OPJS University, Churu,
Rajasthan (India)
Contact No. +91-9992845999
Email- preetyyadav0066@gmail.com

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7/25/2020