**Characterization Of Statistical R-Convergence**

\*Dr. Rajeev Kumar and \*\*Preety

\*Associate Professor, Department of Mathematics, OPJS University, Churu, Rajasthan (India)

\*\*Research Scholar, Department of Mathematics, OPJS University, Churu, Rajasthan (India)

Email: [preetyyadav0066@gmail.com](mailto:preetyyadav0066@gmail.com)

**Abstract:** Convergence of random variables (sometimes called stochastic convergence) is where a set of numbers settle on a particular number. It works the same way as convergence anywhere else; For example, cars on a 5-line highway might converge to one specific lane if there’s an accident closing down four of the other lanes. In the same way, a sequence of numbers (which could represent cars or anything else) can [converge](https://calculushowto.com/converge/) (mathematically, this time) on a single, specific number. Certain processes, distributions and events can result in convergence— which basically mean the values will get closer and closer together. When [Random variables](https://www.statisticshowto.datasciencecentral.com/random-variable/) converge on a single number, they may not settle exactly that number, but they come very, very close. In notation, x (xn → x) tells us that a sequence of random variables (xn) converges to the value x.

[Kumar, R. and Preety. **Characterization Of Statistical R-Convergence.** *Researcher* 2020;12(7):39-42]. ISSN 1553-9865 (print); ISSN 2163-8950 (online). <http://www.sciencepub.net/researcher>. 8. doi:[10.7537/marsrsj120720.08](http://www.dx.doi.org/10.7537/marsrsj120720.08).

**Keywords:** Characterization, Statistical, Convergence

**Introduction**

Statistics is concerned with the collection and analysis of data and with making estimations and predictions from the data. Typically two branches of statistics are discerned: descriptive and inferential. Inferential statistics is usually used for two tasks: to estimate properties of a population given sample characteristics and to predict properties of a system given its past and current properties. To do this, specific statistical constructions were invented. The most popular and useful of them are the average or mean (or more exactly, arithmetic mean) m and standard deviation s (variance s 2). To make predictions for future, statistics accumulates data for some period of time. To know about the whole population, samples are used. Normally such inferences (for future or for population) are based on some assumptions on limit processes and their convergence. Iterative processes are used widely in statistics. For instance the empirical approach to probability is based on the law (or better to say, conjecture) of big numbers, states that a procedure repeated again and again, the relative frequency probability tends to approach the actual probability. The foundation for estimating population parameters and hypothesis testing is formed by the central limit theorem, which tells us how sample means change when the sample size grows. In experiments, scientists measure how statistical characteristics (e.g., means or standard deviations) converge (cf., for example, [23, 31]). Convergence of means/averages and standard deviations have been studied by many authors and applied to different problems (cf. [1-4, 17, 19, 20, 24-28, 35]). Convergence of statistical characteristics such as the average/mean and standard deviation are related to statistical convergence as we show in this section.

Let m and c be the spaces of all bounded and convergent real sequences x = (xk) normed by x = supn |xn|, respectively. Let B be the class of (necessarily continuous) linear functionals β on m which are nonnegative and regular, that is, if x ≥ 0, (i.e., xk ≥ 0 for all k ∈ N:= {1, 2,...}) then β(x) ≥ 0, and β(x) = limk xk, for each x ∈ c. If β has the additional property that β(σ(x)) = β(x) for all x ∈ m, where σ is the left shift operator, defined by σ(x1, x2,...)=(x2, x3,...) then β is called a Banach limit. The existence of Banach limits has been shown by Banach [2,17,19], and another proof may be found in [3]. It is well known [21] that the space of all almost convergent sequences can be represented as the set of all x ∈ m which have the same value under any Banach limit. In the research, we study some generalized limits so that the space of all bounded statistically convergent sequences can be represented as the set of all bounded sequences which have the same value under any such limit. It is proved that the set of such limits and the set of Banach limits are distinct but their intersection is not empty.

In this section we study a useful characterization of statistical r-convergence and some more results.

**Theorem 5.5.1.** A sequence x = {ξk} is statistically r-convergent if and only if every statistically dense subsequence of it is statistically r-convergent.

**Proof.** First suppose that st-r-lim ξk = ξ. Let us take a statistically dense subsequence y =  of x and assume that it is statistically r-divergent.

Then for any real number ξ, there is some ε > 0 such that

δ(Br,ε) > 0

where Br,ε = {kn ∈ ℕ: |– ξ| > r + ε}.

As y is a subsequence of x, we have

Ar,ε ⊇ Br,ε

where Ar,ε = {k ∈ ℕ: |ξk – ξ| > r + ε}.

Consequently, δ(Ar,ε) ≥ δ(Br,ε) > 0

as the subsequence y is statistically dense in x.

This contradicts the fact that x is statistically r-convergent.

Hence y is also statistically r-convergent.

Conversely, suppose that every statistically dense subsequence of x is statistically r-convergent. Then x is also statistically r-convergent since x is a statistically dense subsequence of itself.

This completes the proof of the theorem.

**Corollary 5.2.2.** A statistically r-convergent sequence contains not only dense statistically r-convergent subsequences, but also dense r-convergent subsequences.

**Theorem 5.2.5.** A sequence x = {ξk} is statistically r-convergent ξ to if and only if there exists a set K = {k1 < k2 <…< kn <…} ⊆ ℕ such that δ(K) = 1 and r-lim = ξ.

**Proof.** First suppose that st-r-lim ξk = ξ.

Consider the sets Kr,j = {k ∈ ℕ: |ξk – ξ| < r + } for all j = 1,2,3….

As Kr,j = ℕ – {k ∈ ℕ: |ξk – ξ| ≥ r + } and x is statistically r-convergent to ξ, we have

δ(Kr,j) = 1 j = 1,2,3… ... (1)

Now

Kr,j+1 = {k ∈ ℕ: |ξk – ξ| < r + }

⊂ {k ∈ ℕ: |ξk – ξ| < r + }

= Kr,j.

So

Kr,j+1 ⊂ Kr,j for all j = 1,2,3… …(2)

Let us choose an arbitrary number v1 ∈ Kr,1. Then according to (1) and (2), ∃ v2 > v1, v2 ∈ Kr,2 such that

|{k ≤ n: |ξk – ξ| < r + }| >  for all n ≥ v2.

In a similar way, ∃ v3 > v2, v3 ∈ Kr,3 such that

|{k ≤ n: |ξk – ξ| < r + }| >  for all n ≥ v5.

We continue this process and construct by induction a sequence

v1 < v2 <…< vj <…

of positive integers such that for j = 1,2,3,…

vj ∈ Kr,j and

|{k ≤ n: |ξk – ξ| < r + }| > for all

n ≥ vj. …(3)

Now we construct the set K as follows:

K = {k ∈ ℕ: 1 ≤ k ≤ v1} ∪ (∪ {k ∈ Kr,j: vj ≤ k ≤ vj+1})… (4)

j∈ℕ

Then from (2), (3) and (4) we conclude that for all n from the interval vj ≤ n ≤ vj+1 and for all j = 1,2,3,…, we have

|{k ≤ n: k ∈ K}| = |{k ≤ n: |ξk – ξ| < r + }| > .

Hence it follows that δ(K) = 1. Take some ε > 0 and choose a number j ∈ ℕsuch that  < ε. If n ∈ K and n ≥ vj, then, by definition of K, there exists a number m ≥ j such that vm ≤ n ≤ vm+1 and thus n ∈ Kr,m. Hence we have

|ξn – ξ| < r +  < r + ε.

As this is true for all n ∈ K, we see that r-ξk = ξ.

Conversely, suppose that there exists a set K = {k1 < k2 <…< kn <…} ⊆ ℕ such that δ(K) = 1 and r-lim = ξ. Then for given ε > 0 there is a number n such that for each k ∈ K

|– ξ| < r + ε ∀ k ≥ n. …(5)

Put Ar,ε = {k ∈ ℕ: |– ξ| ≥ r + ε}.

Then we have

Ar,ε ⊆ ℕ – .

Since δ(K) = 1, we get δ(ℕ – ) = 0.

Thus δ(Ar,ε) = 0 for each ε > 0.

⇒ st-r-lim ξk = ξ.

This completes the proof of the theorem.

**Corollary 5.2.5.** A sequence x = {ξk} is statistically r-convergent to ξ if and only if there exists a sequence y = {ηk} such that δ({k ∈ ℕ: ηk = ξk}) = 1 and r-lim ηk = ξ.

**Corollary 5.2.5.** The following statements are equivalent:

1. st-r-lim ξk = ξ;
2. There is a set K = {k1 < k2 <…< kn <…} ⊆ ℕ such that δ(K) = 1 and r-lim ξk = ξ;
3. For each ε > 0, there exists a set K ⊆ ℕ and a number m ∈ K such that δ(K) = 1 and |ξk – ξ| < r + ε for all k ∈ K and k ≥ m.

**Notation.** We denote the set of all statistical r-limits of a sequence x = {ξk} by Lr-st (x), i.e.

Lr-st (x) = {ξ ∈ **R**: st-r-lim ξk = ξ}

**Theorem 5.2.6.** For every sequence x = {ξk} and number r ≥ 0, Lr-st (x) is a convex subset of real numbers.

**Proof.** Let β,η ∈ Lr-st (x) such that β < η and ξ ∈ [β,η]. Then it is enough to prove that ξ ∈ Lr-st (x).

Since ξ ∈ [β,η], there is a number λ ∈ [0,1] such that ξ = λβ + (1–λ)η.

As β,η ∈ Lr-st (x), for each ε > 0 there exist index sets K1, K2 with δ(K1) = δ(K2) =1 and positive integers n1,n2 such that

|ξk – β| < r + ε for all k ∈ K1 and k ≥ n1

|ξk – η| < r + ε for all k ∈ K2 and k ≥ n2.

Let us put K = K1∩K2 and n = max{n1,n2}. Then, since intersection of two statistically dense sets is a statistically dense set, we have δ(K) = 1.

Now for all k ≥ n with k ∈ K, we get

|ξk – ξ| = |ξk – λβ – (1–λ)η|

= |ξk + λξk – λξk – λβ – (1–λ)η|

= |(λξk – λβ) + {(1–λ)ξk – (1–λ)η}|

= |λ(ξk – β) + (1–λ) (ξk – η)|

≤ λ |ξk – β| + (1–λ)|ξk – η|

< λ(r + ε) + (1–λ) (r + ε)

= r + ε

So we conclude from Theorem 5.2.3 that st-r-lim ξk = ξ.

⇒ ξ ∈ Lr-st (x).

Hence Lr-st (x) is a convex subset of real numbers.

This completes the proof of the theorem.

**Lemma 5.2.7.** If q > r, then Lr-st (x) ⊆ Lq-st (x).

**Proof.** Let ξ ∈ Lr-st (x). Then st-r-lim ξk = ξ.

Now by Lemma 5.1.4, st-q-lim ξk = ξ,

i.e. ξ ∈ Lq-st (x).

Hence Lr-st (x) ⊆ Lq-st (x).

This completes the proof of the lemma.

Let x = {ξk} and y = {ηk} be two sequences. Then their sum x + y is equal to the sequence {ξk + ηk} and their difference x – y is equal to the sequence {ξk – ηk}.

**Theorem 5.2.8.** Let st-r-lim ξk = ξ and st-q-lim ηk = η. Then

1. st-(r + q)-lim {ξk + ηk} = ξ + η;
2. st-(r + q)-lim {ξk – ηk} = ξ – η;
3. st-(|c| r)-lim cξk = cξ for any c ∈ **R**

where cx = {cξk}.

**Proof. 1.** Since st-r-lim ξk = ξ, for every ε > 0 there exists a set K1 ⊆ ℕ and a number m1 ∈ K1 such that δ(K1) = 1 and

|ξk – ξ| < r +  ∀ k ∈ K1 and k ≥ m1.

Also st-q-lim ηk = η, then for every ε > 0 there exists a set K2 ⊆ ℕ and a number m2 ∈ K2 such that δ(K2) = 1 and

|ηk – η| < q +  ∀ k ∈ K2 and k ≥ m2.

Let m = max {m1, m2} and K = K1∪K2. Then δ(K) = 1 and ∀ k ∈ Kand k ≥ m, we have

|(ξk + ηk) – (ξ + η)| = |(ξk – ξ) + (ηk – η)|

≤ |ξk – ξ| + |ηk – η|

≤ r +  + q + 

= r + q + ε.

So by Theorem 5.2.3, we have

st-(r + q)-lim {ξk + ηk} = ξ + η.

**2.** From part (1), ∀ k ∈ Kand k ≥ m, we have

|(ξk – ηk) – (ξ – η)| = |(ξk – ξ) – (ηk – η)|

≤ |ξk – ξ| + |ηk – η|

≤ r +  + q + 

= r + q + ε.

So by Theorem 5.2.3, we have

st-(r + q)-lim {ξk – ηk} = ξ – η.

**5.** Since st-r-lim ξk = ξ, for every ε > 0 there exists a set K ⊆ ℕ and a number m ∈ Ksuch that δ(K) = 1 and

|ξk – ξ| < r +  ∀ k ∈ Kand k ≥ m.

Now

|cξk – cξ| = |c||ξk – ξ|

< |c| (r + )

= |c|r + ε.

So by Theorem 5.2.3, we have

st-(|c| r)-lim cξk = cξ.

**Corollary 5.2.9.** If st-lim ξk = ξ and st-lim ηk = η. Then

1. st-lim {ξk + ηk} = ξ + η;
2. st-lim {ξk – ηk} = ξ – η;
3. st-lim cξk = cξ for any c ∈ **R**.

**Corresponding author:**

Mrs. Preety

Research Scholar, Department of Mathematics,

OPJS University, Churu,

Rajasthan (India)

Contact No. +91-9992845999

Email- preetyyadav0066@gmail.com

**References:**

1. I.J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Camb. Phil. Soc.* 104(1988) 141-145.
2. Mursaleen, Invariant means and some matrix transformations, *Tamkang J. Math.* 10(2) (1979) 183-188.
3. Mursaleen, Matrix transformations between some new sequence spaces, *Houston J. Math.* 9 (4) (1983) 505-509.
4. Mursaleen, λ-statistical convergence, *Math. Slovaca* 50 (1) (2000) 111-115.
5. I. Niven and H.S. Zuckerman, *An Introduction to The Theory of Numbers*, Fourth Ed., New York, John Wiley and Sons, 1980.
6. T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* 30 (2) (1980) 139-150.
7. E. Savas, Some sequence spaces involving invariant means, *Indian J. Math.* 31 (1) (1989) 1-8.
8. E. Savas, On lacunary strong σ-convergnce, *Indian J. Pure Appl. Math.* 21 (4) (1990) 359-365.
9. E. Savas and F. Nuray, On σ-statistical convergence and lacunary σ-statistical convergence, *Math. Slovaca* 43 (1993) 309-315.
10. E. Savas, Strong almost convergence and almost λ-statistical convergence, *Hokkaido Math. J.* 29 (2000) 531-536.

7/25/2020