Numerical Range of Self-Inverse Matrices

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Abstract: For an $n \times n$ matrix A, let $M(A)$ be the smallest possible constant in the inequality, $D_P(A) \leq M(A)R_P(A)$. Here, P is a point on the smooth portion of the boundary $\partial W(A)$ of the numerical range. $A, R_P(A)$ is the radius of curvature of $\partial W(A)$ at this point, and $D_P(A)$ is the distance from P to the spectrum of A. In this paper we compute the $M(A)$ for matrix A which is self-inverse. 200 Mathematics Subject Classification. Primary 47A12; Secondary 15A42, 14H50.

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1. Introduction

Let A be an $n \times n$ matrix with complex entries: $\mathbb{C}^{n\times n}$. The numerical range of A is defined as

$$W(A) = \{< Ax, x >: x \in \mathbb{C}^n, ||x|| = 1\}.$$

where $<.,.>$ and $||.||$ are the standard scalar product and norm on $\mathbb{C}^n$, respectively.

Thus the numerical range of an operator, like spectrum, is a subset of the complex plane whose geometrical properties should say something about the operator.

2. Preliminaries and Auxiliary Results

One of the most fundamental properties of the numerical range is its convexity, stated by the famous Toepliz-Hausdorff Theorem. Other important property of $W(A)$ is that its closure contains the spectrum of the operator. $W(A)$ is a connected set with a piecewise analytic boundary $\partial W(A)$ for details see [2].

Hence, for all but finitely many point $P \in \partial W(A)$, the radius of curvature $R_P(A)$ of $\partial W(A)$ at P is well defined. By convention, $R_P(A) = 0$ if P is a corner point of $W(A)$, and $R_P(A) = \infty$ if P lies inside a flat portion of $\partial W(A)$. Let $D_P(A)$ denote the distance from P to $\sigma(A)$, we define $M(A)$ the smallest constant such that

$$D_P(A) \leq M(A)R_P(A) \quad (1)$$

for all $P \epsilon \partial W(A)$, where $R_P(A)$ is defined.

By Donoghes theorem $D_P(A) = 0$ whenever $R_P(A) = 0$. Therefor, $M(A) = 0$ for all convexoid element A. Recall that convexoid element is an element such that its numerical range coincides with the convex hull of its spectrum. For non-convexoid A,

$$M(A) = sup \frac{D_P(A)}{R_P(A)} (2)$$

where the supremum in the right-hand side is taken along all points $P \epsilon \partial W(A)$ with finite non-zero curvature. Computation of $M(A)$ for arbitrary A is an interesting problem. Computation of $M(A)$ for arbitrary $n \times n$ matrix A is also an interesting open problem.

For we $n > 3$, do not have an exact value of

$$M_n = \sup\{M(A): A \epsilon \mathbb{C}^{n\times n}\}$$

the question whether there exists a universal constant $M = sup_n M_n$, posed by Mathias [4].

In [1] the authors have proved that

$$\frac{n}{2}\sin\left(\frac{\pi}{n}\right) \leq M_n \leq \frac{n}{2}. \ (3)$$

In [5] the author find a sequence of $n \times n$ Toeplitz nilpotent matrices A_n with $M(A_n)$ algorowing asymptotically as $\log n$.

Hence, the ansewer to Mathias question is negative.

However, the lower bound in (3) is still of some interest, at least for small values of

The question of the exact rate of growth of M_n (is it $\log n$, or n, or something in between) remains open.

3. Main Results

Let the operator A be self-inverse, i.e., $A^2 = I$ but $A \neq \pm I$, so $\sigma(A) = \{\pm 1\}$.

Also $\partial W(A)$ is an ellipse with foci at ± 1 and major/minor axis $||A|| \pm \frac{1}{||A||}$ [3].

If $\partial W(A) = a\cos(\theta) + ibsin(\theta)$ with $a^2 = b^2 + 1$ then

$$M(A) = \max\left\{\frac{\sqrt{a^2-1}}{a}, \frac{a}{a+1}\right\}.$$

Then we have following main Theorem:

Let the operator be non trivial self-inverse, then

$$M(A) = \max\left\{\frac{||A||^2-1}{||A||^2+1}, \frac{||A||^2+1}{(||A||+1)^2}\right\}.$$

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