A study of b-vex function with convex optimization

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Abstract: In this paper we deal with different generalizations of b-vex functions. Some known properties of convex functions are studied for b-vex functions. Stress is given to establish certain interrelations among these functions. These results are also extended for vector valued functions on the lines of work done by **[8]**.
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Introduction

The class of convex functions has been extended to a class of b-vex functions by [1], which are quite similar to (α, λ)-concave functions introduced by [2]. We now define b-vex functions and study some of their properties.

Let X be a non-empty convex subset of R^n, and let R_+ denote the set of non-negative real numbers. Let $f: X \to R, b: X \times X \times [0,1] \to R_+$. We now define b-vex functions on the lines of [1] by taking $b_1(x, u, \lambda) = \lambda\, b(x, u, \lambda)$ and $b_2\,(x, u, \lambda) = 1 - \lambda b(x, u, \lambda)$, where $b_1(x, u, \lambda) \geq 0$ and $b_2(x, u, \lambda) \geq 0$.

Definition 1 : The function f is said to be b-vex at each $u \in X$. If there exists a function $b(x, u, \lambda)$ such that, for $x \in X,\ 0 \leq \lambda \leq 1$

$$f\,[\lambda x + (1-\lambda)u] \leq \lambda b(x, u, \lambda)\, f(x) + (1 - \lambda b(x, u, \lambda)]\, f(u)$$

where $\lambda b(x, u, \lambda) \leq 1$. f is said to be b-vex on X if it is v-vex at each $u \in X$. For the sake of brevity, the argument of b is omitted unless needed for specification.

Definition 2 : Given $S \subseteq R^n \times R$, S is said to be b-vex set if $(x, \alpha), (u, \beta) \in S$ imply that for $0 \leq \lambda \leq 1$

$$(\lambda x + (1-\lambda)\, u,\ \lambda b\alpha + (1-\lambda b)\beta) \in S$$

A characterization of b-vex functions is now presented in terms of b-vexity of the epigraph E(f), where

$$E(f) = \{(x, \alpha) \mid x \in X,\ \alpha \in R,\ f(x) \leq \alpha\}$$

Theorem 1 : A numerical function f defined on a convex subset X of R^n is b-vex if and only if E(f) is a b-vex set in $R^n \times R$.

Proof : Suppose that f is b-vex on X. Let $(x, \alpha), (u, \beta) \in E(f)$. Then $f(x) \leq \alpha$ and $f(u) \leq \beta$. Since f is b-vex on X, for $0 \leq \lambda \leq 1$,

$$\begin{aligned} f(\lambda x + (1-\lambda)u) &\leq \lambda b f(x) + (1-\lambda b)\, f(u) \\ &\leq \lambda b\alpha + (1-\lambda b)\, \beta \end{aligned}$$

Thus, for $0 \leq \lambda \leq 1$,

$$(\lambda x + (1-\lambda)u,\quad \lambda b\alpha + (1-\lambda b)\beta \in E(f)$$

Hence, E(f) is a b-vex set.

Conversely, assume that E(f) is a b-vex set. Let $x, u \in X$. Then $(x, f(x)) \in E(f)$, $(u, f(u)) \in E(f)$. Thus, for $0 \leq \lambda \leq 1$.

$$(\lambda x + (1-\lambda)\, u,\ \lambda b f(x) + (1-\lambda b)\, f(u)) \in E(f)$$

and it follows that, for $0 \le \lambda \le 1$

$$f(\lambda x + (1-\lambda)u) \le \lambda b f(x) + (1-\lambda b) f(u)$$

Hence, f is a b-vex function on X.

Theorem 2 : If $(S_i)_{i \in I}$ is a family of b-vex sets in $R^n \times R$, then their intersection $\bigcap_{i \in I} S_i$ is also a b-vex set.

Proof : Let $(x, \alpha), (u, \beta) \in \bigcap_{i \in I} S_i$ and let $0 \le \lambda \le 1$. Then for each $i \in I, (x, \alpha), (u, \beta) \in S_i$. Since S_i is a b-vex set, for each $i \in I$, it follows that

$$(\lambda x + (1-\lambda)u,\ \lambda b\alpha + (1-\lambda b)\beta) \in S_i$$

Thus, for $0 \le \lambda \le 1$

$$(\lambda x + (1-\lambda)u,\ \lambda b\alpha + (1-\lambda b)\beta) \in \bigcap_{i \in I} S_i$$

Hence, the result follows.

Theorem 3 : If $(f_i)_{i \in I}$ is a family of numerical functions which are b-vex and bounded from above on a convex set X in R^n, then the numerical function

$f(x) = \sup_{i \in I} f_i(x)$ is a b-vex function on X.

Proof : Since each f_i is a b-vex function on X, its epigraph

$$E(f_i) = \{(x, \alpha) \mid x \in X,\ \alpha \in R,\ f_i(x) \le \alpha\}$$

is a b-vex set in $R^n \times R$. Therefore, their intersection

$$\bigcap_{i \in I} E(f_i) = \{(x, \alpha) \mid x \in X,\ \alpha \in R,\ f_i(x) \le \alpha,\ i \in I\}$$

$$= \{(x, \alpha) \mid x \in X,\ \alpha \in R,\ f(x) \le \alpha\}$$

is also a b-vex set in $R^n \times R$, by Theorem 2. This intersection is the epigraph of f. Hence, by Theorem 1, f is a b-vex function on X.

A convex function defined on some open set is a continuous function [6]. But it can be seen from the following example that it is not necessarily true for a b-vex function.

Example 1 : Let X =]0, 2[. Define a function $f : X \to R$ by

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ x & \text{if } 1 \le x < 2 \end{cases}$$

Let $b : X \times X [0,1] \to R_+$ be defined by

$$b(x, u, \lambda) = \begin{cases} \lambda(u-x)/u & \text{if } x \le u \\ (u + \lambda(x-u))/\lambda x & \text{if } x > u, \lambda \ne 0 \\ 1 & \text{if } x > u, \lambda = 0 \end{cases}$$

It can be seen that f is a b-vex function and epigraph E(f) of f is a b-vex set in R x R, but f is not continuous at x = 1.

Conclusion

Every convex function is b-vex, where $b(x, u, \lambda) \equiv 1$. However, the converse is not necessarily true. The function f considered in Example 1 is b-vex but is not convex, because for x = 1/2, u = 3/2, $\lambda = 1/2$,

$$f(\lambda x + (1-\lambda)u) > \lambda f(x) + (1-\lambda) f(u).$$

References

[1]. C. R. Bector and C. Singh, B-vex Functions, Journal of Optimization Theory and Applications, 71(2), 237-253(1991).

[2]. E. Castagnoli and P. Mazzoleni, About Derivatives of Some Generalized Concave Functions, Continuous-Time Fractional and Multiobjective Programming, edited by C. Singh and B. K. Dass, Analytic Publishing Company, New Delhi, 53-63(1989).

[3]. Goran Lesaja And Verlynda N. Slaughter, Interior-Point Algorithms For A Class Of Convex Optimization Problems, Yugoslav Journal Of Operations Research Volume 19 Number 2, 239-248(2009).

[4]. J. M. Borwein and A. S. Lewis. Convex Analysis and Nonlinear Optimization. Springer, 2000.

[5]. M.P. Bendsøe, A. Ben-Tal and J. Zowe, "Optimization methods for truss geometry and topology design," Structural Optimization, vol. 7, pp. 141–159, (1994).

[6]. O. L. Mangasarian, Non-linear Programming, McGraw-Hill, New York(1969).

[7]. Stephen Boyd and Lieven Vandenberghe, " Convex Optimization", cambridge university press.

[8]. V. Jeyakumar and B. Mond, On Generalized Convex Mathematical Programming, Journal of Australian Mathematical Society, Ser. (B), 34(1), 43-53 (1992).

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