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Solving Korteweg de-Vries (KdVs) Equations of Fractional-order using Modified Riemann-Liouville Derivative

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Abstract: This paper witnesses the combination of an efficient transformation and Exp-function method to construct generalized solitary wave solutions of the nonlinear Korteweg de-Vries (KdVs) equations of fractional-order. Computational work and subsequent numerical results re-confirm the efficiency of proposed algorithm. It is observed that suggested scheme is highly reliable and may be extended to other nonlinear differential equations of fractional order.
[Qazi Mahmood Ul Hassan , Syed Tauseef Mohyud-Din.**Solving Korteweg de-Vries (KdVs) Equations of Fractional-order using Modified Riemann-Liouville Derivative**. *Rep Opinion* 2023;15(8):1-6]. ISSN 1553-9873 (print); ISSN 2375-7205 (online). http://www.sciencepub.net/report. 01.doi:10.7537/marsroj150823.01.

Keywords: Korteweg de-Vries (KdVs) equations , fractional calculus, exp-function method, modified Riemann-Liouville derivative.

1. Introduction

The subject of factional calculus [1, 2] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics. In recent years, nonlinear fractional differential equations (NFDEs) have gained much interest due to exact description of nonlinear phenomena of many real-time problems. The fractional calculus is also considered as a novel topic [3, 4]; has gained considerable popularity and importance during the recent past. It has been the subject of specialized conferences, workshops and treatises or so, mainly due to its demonstrated applications in numerous seemingly diverse and widespread fields of science and engineering. Some of the areas of present-day applications of fractional models [5-8] include fluid flow, solute transport or dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, material viscoelastic theory, electromagnetic theory, dynamics of earthquakes, control theory of dynamical systems, optics and signal processing, bio-sciences, economics, geology, astrophysics, probability and statistics, chemical physics, and so on. As a consequence, there has been an intensive development of the theory of fractional differential equations, see [1–8] and the references therein. Recently, He and Wu [9] developed a very efficient technique which is called exp-function method for solving various nonlinear physical problems. The through study of literature reveals that Exp-function method has been applied on a wide range of differential equations and is highly reliable. The exp-function method has been extremely useful for diversified nonlinear problems of physical nature and has the potential to cope with the versatility of the complex nonlinearities of the problems. The subsequent works have shown the complete reliability and efficiency of this algorithm. He et. al. [10-11] used this scheme to find periodic solutions of evolution equations; Mohyud-Din [12-15] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [16] tried this novel approach for Fisher's equation; Wu et. al. [17, 18] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [19] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [21, 22] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [23] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [24] for an explicit and numerical solutions of the fractional KdV equation; The basic motivation of this paper is the development of an efficient combination comprising an efficient transformation**,** exp-function method using Jumarie's derivative approach [25-28] and its subsequent application to construct generalized solitary wave solutions of the nonlinear Korteweg de-Vries (KdVs) Equations of fractional-order [29-30].

It is to be highlighted that Ebaid [31] proved that $c=d$ and $p=q$ are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation.

Theorm.1.1 [31] suppose that $u^{(r)}$ and u^{γ} are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and γ are both positive integers. Then the balancing procedure using the Exp-function ansatz; $U(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp(n\eta)}{\sum_{m=-p}^{q} b_m \exp(m\eta)}$, leads to $c=d$ and $p=q$, $\forall r,s,\Omega,\lambda \geq 1$.

Theorm.1.2 [31] suppose that $u^{(r)}$ and $u^{(s)}u^{k}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r,s,k \geq 1$.

Theorm 1.3[31] Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s and Ω are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall r,s \geq 1, \forall \Omega \geq 2$.

Theorm 1.4[31] Suppose that $u^{(r)}$ and $(u^{(s)})^{\Omega}u^{\lambda}$ are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r,s,Ω and λ are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to $c=d$ and $p=q$, $\forall\, r,s,\Omega,\lambda \geq 1$.

2. Jumarie's Fractional Derivative

Jumarie's fractional derivative is a mo1dified Riemann-Liouville derivative defined as

$$D_x^{\alpha} f(x)=\begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^x (x-t)^{-\alpha-1}(f(t)-f(0))dt, & \alpha \leq 0, \\ \frac{1}{\Gamma(-\alpha)}\frac{d}{dx}\int_0^x (x-t)^{-\alpha}(f(t)-f(0))dt, & 0 \leq \alpha \leq 1, \\ \left[f^{\alpha-n}(x)\right]^{n}, & n \leq \alpha \leq n+1, n \geq 1 \end{cases} \quad (1)$$

Where $f:\mathbb{R}\to\mathbb{R}, x\to f(x)$ denotes a continuous (but not necessarily differentiable) function.

Some useful formulas and results of Jumarie's modified Riemann–Liouville derivative were summarized in Refs. [25-28].

$$D_x^{\alpha} c=0, \alpha \geq 0, \ c=\text{constant} \quad (2)$$

$$D_x^{\alpha}[cf(x)]=cD_x^{\alpha} f(x), \alpha \geq 0, \ c=\text{constant} \quad (3)$$

$$D_x^{\alpha} x^{\beta}=\frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha}, \beta \geq \alpha \geq 0. \quad (4)$$

$$D_x^{\alpha}[f(x)g(x)]=[D_x^{\alpha} f(x)]g(x)+f(x)[D_x^{\alpha} g(x)], \quad (5)$$

$$D_x^{\alpha} f(x(t))=f_x'(x)x^{\alpha}(t), \quad (6)$$

3. Exp-function Method [32-36]

We consider the general nonlinear FPDE of the type

$$P(u,u_t,u_x,u_{xx},u_{xxx},\ldots,D_t^{\alpha}u,D_x^{\alpha}u,D_{xx}^{\alpha}u,\ldots)=0, \ 0<\alpha \leq 1, \quad (7)$$

where $D_t^{\alpha}u, D_x^{\alpha}u, D_{xx}^{\alpha}u$ are the modified Riemann-Liouville derivative of u with respect to t,x,xx respectively.

Using a transformation [36]

$$\eta=kx+\frac{\omega t^{\alpha}}{\Gamma(1+\alpha)}+\eta_0, \quad (8)$$

k,ω,η_0 are all constants with $k,\omega \neq 0$

we can rewrite equation (1) in the following nonlinear ODE;

$$Q(u,u',u'',u''',u^{iv})=0, \quad (9)$$

where the prime denotes derivative with respect to η.

According to Exp-function method, we assume that the wave solution can be expressed in the following form

$$u(\eta)=\frac{\sum_{n=-c}^{d} a_n \exp[n\eta]}{\sum_{m=-p}^{q} b_m \exp[m\eta]} \quad (10)$$

where p,q,c and d are positive integers which are known to be further determined, a_n and b_m are unknown constants. We can rewrite Eq.(4) in the following equivalent form

$$u(\eta)=\frac{a_c \exp(c\eta)+\ldots+a_{-d}\exp(-d\eta)}{b_p \exp(p\eta)+\ldots+b_{-q}\exp(-q\eta)}. \quad (11)$$

This equivalent formulation plays an important and fundamental part for finding the analytic solution of problems. To determine the value of c and p by using [31],

$$p = c, q = d \quad (12)$$

4. Numerical Applications

In this section, we apply exp-function method to construct generalized solitary solutions for Korteweg de-Vries(KdVs) equations of fractional-order. Numerical results are very encouraging.

Example 4.1 Consider the following KdVs Equation of fractional order

$$D_t^{\alpha} u + 6uu_x = u_{xxx}, \quad 0 < \alpha \leq 1 \quad (13)$$

Using (8) equation (13) can be converted to an ordinary differential equation

$$wu' + 6kuu' = k^2 u''. \quad (14)$$

where the prime denotes the derivative with respect to η. The solution of the equation (13) can be expressed in the form, equation (11). To determine the value of c and p, by using [31],

$$p = c, q = d. \quad (15)$$

Case 4.1.I. We can freely choose the values of c and d, but we will illustrate that the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $q = d = 1$ equation (11) reduces to

$$u(\eta) = \frac{a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]}{b_1 \exp[\eta] + a_0 + b_{-1} \exp[-\eta]}. \quad (16)$$

Substituting equation (16) into equation (14), we have

$$\frac{1}{A}\left[\begin{array}{l} c_4 \exp(4\eta) = c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) \\ + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \\ + c_{-3} \exp(-3\eta) + c_{-4} \exp(-4\eta) \end{array}\right] = 0, \quad (17)$$

where $A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^4$, c_i are constants obtained by Maple 16. Equating the coefficients of $\exp(n\eta)$ to be zero, we obtain

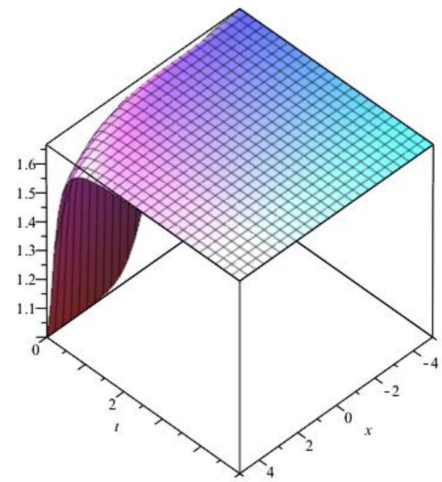
$$\{c_{-4} = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0\} \quad (18)$$

Solution of (12) will yield

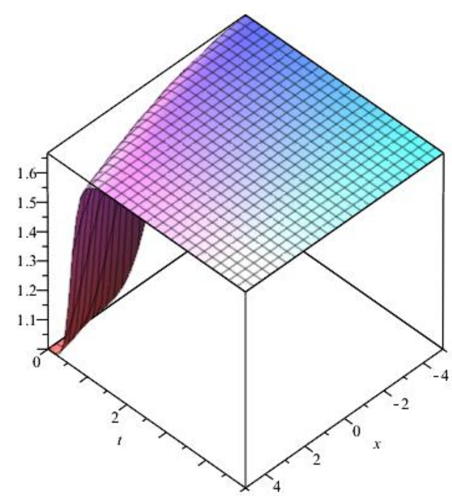
$$\left\{a_{-1} = \frac{1}{3}\frac{b_{-1}(2kb_1 + 3a_1)}{b_1}, \omega = -\frac{2k(kb_1 + 3a_1)}{b_1}, a_0 = 0, a_1 = a_1, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1\right\}. \quad (19)$$

We, therefore, obtained the following generalized solitary solution $u(x,t)$ of equation (13)

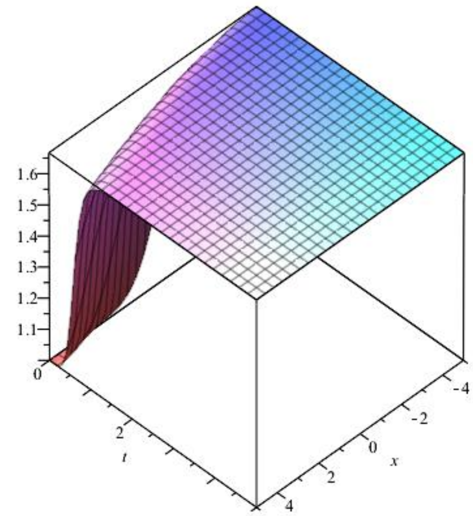
$$u(x,t) = \left\{ \frac{\dfrac{\frac{1}{3}\frac{b_{-1}(2kb_1 + 3a_1)}{b_1} e^{-\left(xk - \frac{2k(kb_1+3a_1)}{b_1}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)}}{k} + \dfrac{a_1 e^{xk - \frac{2k(kb_1+3a_1)}{b_1}\frac{t^{\alpha}}{\Gamma(1+\alpha)}}}{k}}{b_{-1} e^{-\left(xk - \frac{2k(kb_1+3a_1)}{b_1}\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right)} + b_1 e^{xk - \frac{2k(kb_1+3a_1)}{b_1}\frac{t^{\alpha}}{\Gamma(1+\alpha)}}} \right\} \quad (20)$$



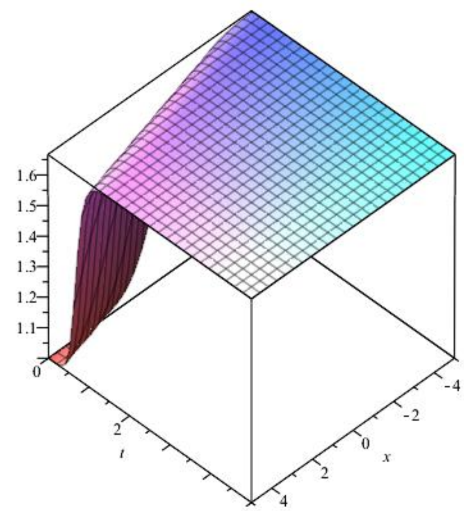
4.1(a) $\alpha = .25$



4.1(b) $\alpha = .50$



4.1(c) $\alpha=.75$



4.1(d) $\alpha=1$

Figs 4.1(a), 4.1(b), 4.1(c), 4.1(d) Soliton solutions of equation (13) for $a_0=b_1=b_{-1}=\omega=1$ and $k=1$

Case 4.1.II. If $p=c=2$ and $q=d=1$ then trial solution, equation (13) reduces to

$$u(\eta)=\frac{a_2\exp[2\eta]+a_1\exp[\eta]+a_0+a_{-1}\exp[-\eta]}{b_2\exp[2\eta]+b_1\exp[\eta]+b_0.+b_{-1}\exp[-\eta]}. \quad (21)$$

Proceeding as before, we obtain

$$\left\{a_{-1}=\frac{1}{3}\frac{b_{-1}(2kb_1+3a_1)}{b_1},\omega=-\frac{2k(kb_1+3a_1)}{b_1},a_0=0,a_1=a_1,b_{-1}=b_{-1},b_0=0,b_1=b_1\right\}. \quad (22)$$

Hence we get the generalized solitary wave solution of equation (13) for $\alpha=1$ as follow

$$\left\{\frac{\dfrac{\frac{1}{3}\frac{b_{-1}(2kb_1+3a_1)}{b_1}e^{-\left(xk-\frac{2k(kb_1+3a_1)}{b_1}t\right)}}{k}+\dfrac{a_1e^{\left(xk-\frac{2k(kb_1+3a_1)}{b_1}t\right)}}{k}}{b_{-1}e^{-\left(xk-\frac{2k(kb_1+3a_1)}{b_1}t\right)}+b_1e^{\left(xk-\frac{2k(kb_1+3a_1)}{b_1}t\right)}}\right\} \quad (23)$$

In both cases, for different choices of c, p, d and q, we get the same soliton solutions which clearly illustrate that final solution does not strongly depends upon these parameters.

5. Conclusion In this paper, we applied exp-function method to construct generalized solitary solutions of the nonlinear fractional order Korteweg de-Vries (KdVs) equations. It is observed that the Exp-function method is very convenient to apply and is very useful for finding solutions of a wide class of nonlinear problems.

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References

[1]. R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[2]. A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204, Elsevier, Amsterdam, The Netherlands, 2006.
[3]. K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, A Wiley-Interscience Publication, John Wiley & Sons, New York, NY, USA, 1993.
[4]. I. Podlubny, Fractional Differential Equations, vol. 198 of Mathematics in Science and Engineering, Academic Press, San Diego, Calif, USA, 1999. G. Adomian, A review of the decomposition method in applied mathematics, J. Math. Anal. Appl. Vol.135 (1988) No.2, pp.501-544.

[5]. N. T. Shawagfeh, Analytical approximate solutions for nonlinear fractional differential equations, Appl. Math. Comput. 131 (2-3) (2002) 517-529.

[6]. S. S. Ray and R. K. Bera, An approximate solution of a nonlinear fractional differential equation by Adomian's decomposition method, Appl. Math. Comput., 167, no. 1, 561–571, 2005.

[7] J. H. He, Some applications of nonlinear fractional differential equations and their approximations, Bull. Sci. Technol. 15 (2) (1999) 86–90.

[8]. A. Yildirim and S. T. Mohyud-Din, S. Sarıaydın, Numerical comparison for the solutions of an harmonic vibration of fractionally damped nano-sized oscillator, J. of King Saud Uni – Sci (2011) 23, 205–209.

[9]. J. H. He and X. H. Wu, Exp-function method for nonlinear wave equations, Chaos Solitons & Fractals 30 (3) (2006), 700–708.

[10]. J. H. He, An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering, Int. J. Mod. Phys. B 22 (21) (2008), 3487-4578.578.

[11]. J. H. He and M. A. Abdou, New periodic solutions for nonlinear evolution equation using exp-method, Chaos, Solitons & Fractals, 34 (2007), 1421-1429.

[12]. S. T. Mohyud-Din, M. A. Noor and A. Waheed, Exp-function method for generalized travelling solutions of good Boussinesq equations, J. Appl. Math. Computg. 30 (2009), 439-445, DOI 10.1007/s12190-008-0183-8.

[13]. S. T. Mohyud-Din, M. A. Noor and K. I. Noor, Some relatively new techniques for nonlinear problems, *Mathematical Problems in Engineering, Hindawi*, 2009 (2009); Article ID 234849, 25 pages, doi:10.1155/2009/234849.

[14]. M. A. Noor, S. T. Mohyud-Din and A. Waheed, Exp-function method for solving Kuramoto-Sivashinsky and Boussinesq equations, J. Appl. Math. Computg. 29 (2008), 1-13, DOI: 10.1007/s12190-008-0083-y.

[15]. M. A. Noor, S. T. Mohyud-Din and A. Waheed, Exp-function method for generalized traveling solutions of master partial differential equations, *Acta Applicandae Mathematicae, Springer* 104 (2) (2008), 131–137, DOI: 10.1007/s10440-008-9245-z;

[16]. T. Ozis, C. Koroglu, A novel approach for solving the Fisher's equation using Exp-function method, Phys Lett. A 372 (2008) 3836 - 3840

[17]. X. H. Wu and J. H. He, Exp-function method and its applications to nonlinear equations, Chaos, Solitons & Fractals, (2007), in press.

[18]. X. H. Wu and J. H. He, Solitary solutions, periodic solutions and compacton like solutions using the exp-function method, Comput. Math. Appl. 54 (2007), 966-986.

[19]. E. Yusufoglu, New solitonary solutions for the MBBN equations using exp-function method, Phys. Lett. A. 372 (2008), 442-446.

[20]. S. Zhang, Application of exp-function method to high-dimensional nonlinear evolution equation, Chaos, Solitons & Fractals, 365 (2007), 448-455.

[21]. S. D. Zhu, Exp-function method for the Hybrid-Lattice system, Inter. J. Nonlin. Sci. Num. Simulation, 8(2007), 461-464

[22]. S. D. Zhu, Exp-function method for the discrete m KdV lattice, Inter. J. Nonlin. Sci. Num. Simulation, 8(2007), 465-468.

[23]. N.A. Kudryashov, Exact soliton solutions of the generalized evolution equation of wave dynamics, J. Appl Math and Mech, 52 (3), (1988), 361

[24] S. Momani, An explicit and numerical solutions of the fractional KdV equation, Math. Comput. Simul. 70 (2) (2005) 110-118.

[25]. J. H. He, Z.B. Li, Fractional Complex Transform for Fractional Differential Equations, Math and Comput Appl, 15 (5) (2010) 970-973.

[26]. Z.B. Li, An Extended Fractional Complex Transform, J. of Nonlinear Sc and Numl Simul, 11 (2010) 0335-0337.

[27]. J.H. He, Z.B. Li, Converting Fractional differential equations into partial differential equations, Thermal Sc., DOI REFERENCE: 10.2298/TSCI110503068H.

[28]. G. Jumarie Modified Riemann-Liouville Derivative and Fractional Taylor series of Non-differentiable Functions Further Results, Comput. Math. Appl. 51 (9-10) (2006) 1367-1376.

[29]. M. A. Abdou, A. A. Soliman and S. T. Basyony, New application of exp-function method for improved Boussinesq equation. Phys. Lett. A, 369 (2007), 469-475.

[30]. S. A. El-Wakil, M. A. Madkour and M. A. Abdou, Application of exp-function method for nonlinear evolution equations with variable co-efficient, Phys. Lett. A, 369 (2007), 62-69.

[31]. A. Ebaid An improvement on the Exp-function method when balancing the highest order linear and nonlinear terms J. Math. Anal. Appl. 392(2012) 1–5.

[32]. M. A. Abdou, A. A. Soliman and S. T. Basyony, New application of Exp-function method for improved Boussinesq equation. Phys. Lett. A, 369 (2007), 469-475.

[33]. J. H. He and M. A. Abdou, New periodic solutions for nonlinear evolution equation using exp-method, Chaos Solitons. Farct.34 (2007), 1421-1429.

[34]. M. A. Noor, S. T. Mohyud-Din and A. Waheed, Exp-function method for solving Kuramoto-Sivashinsky and Boussinesq equations, J. Appl. Math. Computg. (2008), DOI: 10.1007/s12190-008-0083-y.

[35]. M. A. Noor, S. T. Mohyud-Din and A. Waheed, Exp-function method for generalized travelling solutions of master partial differential equations, Acta Applnda. Mathmtce. (2008), DOI: 10.1007/s10440-008-9245-z.

[36]. Z. Bin G/G-Expansion Method for Solving Fractional Partial Differential Equations in the Theory of Mathematical Physics, Commun. Theor. Phys. 58 (2012) 623–630.

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