Websites:
http://www.sciencepub.net
http://www.sciencepub.net/report

Emails:
editor@sciencepub.net
reportopinion@gmail.com

Report and Opinion

MARSLAND PRESS
Multidisciplinary Academic Journal Publisher

Studies On σ-Statistical Convergence And Lacunary σ-Statistical Convergence

*Annu Rani and **Dr. Rajeev Kumar

*Research Scholar, Department of Mathematics, SunRise University, Alwar, Rajasthan (India)
**Associate Professor, Department of Mathematics, SunRise University, Alwar, Rajasthan (India)
Email: puniajit50@gmail.com

Abstract: In this paper we study one more extension of the concept of statistical convergence namely almost λ-statistical convergence. In section 1.2 we discuss some inclusion relations between almost λ-statistical convergence, strong almost (V,λ)-summability and strong almost convergence. Further in section 1.3 we study the necessary and sufficient condition for an almost statistically convergent sequence to be almost λ-statistically convergent.

[Rani, A. and Kumar, R. **STUDIES ON σ-STATISTICAL CONVERGENCE AND LACUNARY** σ-STATISTICAL CONVERGENCE. *Rep Opinion* 2021;13(10):1-6]. ISSN 1553-9873 (print); ISSN 2375-7205 (online). http://www.sciencepub.net/report. 1. doi:10.7537/marsroj131021.01.

Keywords: Parameters, Concetroidal matrices, Class 1, Related

1.1 Introduction

Let s be the set of all real or complex sequences and let l_∞, c and c_0 denote the Banach spaces of bounded, convergent and null sequences x = {ξ_k} respectively normed by $||x|| = \sup_k |\xi_k|$. Suppose D is the shift operator on s, i.e. $D(\{\xi_k\}) = \{\xi_{k+1}\}$.

Definition 1.1.1. A Banach limit [1] is a linear functional L defined on l_∞, such that

(i) $L(x) \geq 0$ if $\xi_k \geq 0$ for all k,

(ii) $L(Dx) = L(x)$ for all $x \in l_\infty$,

(iii) $L(e) = 1$ where $e = \{1,1,1,\ldots\}$.

Definition 1.1.2. A sequence $x \in l_\infty$ is said to be almost convergent [19] if all Banach limits of x coincide.

Let $\hat{c}$ and $\hat{c}_0$ denote the sets of all sequences which are almost convergent and almost convergent to zero. It was proved by Lorentz [19] that

$$\hat{c} = \{x = \{\xi_k\}: \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi_{k+m} \text{ exists uniformly in m}\}.$$

Several authors including Duran [7], King [15] and Lorentz [19] have studied almost convergent sequences.

Definition 1.1.3. A sequence x = {ξ_k} is said to be (C,1)-summable if and only if $\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \xi_k$ exists.

Definition 1.1.4. A sequence x = {ξ_k} is said to be strongly (Cesáro) summable to the number ξ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_k - \xi| = 0.$$

Spaces of strongly Cesáro summable sequences were discussed by Kuttner [17] and some others and this concept was generalized by Maddox [20].

Remark 1.1.1. Just as summability gives rise to strong summability, it was quite natural to expect that almost convergence must give rise to a new type of convergence, namely strong almost convergence and this concept was introduced and discussed by Maddox [20].

Definition 1.1.1. A sequence $x = \{\xi_k\}$ is said to be strongly almost convergent to the number ξ if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m.}$$

If $[\hat{c}]$ denotes the set of all strongly almost convergent sequences, then

$$[\hat{c}] = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m}\}.$$

Let $\lambda = \{\lambda_n\}$ be a non-decreasing sequence of positive numbers tending to ∞ such that

$$\lambda_{n+1} \le \lambda_n + 1, \lambda_1 = 1.$$

Definition 1.1.7. Let $x = \{\xi_k\}$ be a sequence. The generalized de la Valée-Pousin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} \xi_k$$

where $I_n = [n - \lambda_n + 1, n]$.

Definition 1.1.8. A sequence $x = \{\xi_k\}$ is said to be (V,λ)-summable to a number ξ [18] if $t_n(x) \to \xi$ as $n\to\infty$.

Remark 1.1.9. Let $\lambda_n = n$. Then $I_n = [1, n]$ and

$$t_n(x) = \frac{1}{n} \sum_{k=1}^{n} \xi_k .$$

Hence (V,λ)-summability reduces to $(C,1)$-summability when $\lambda_n = n$.

Definition 1.1.10. A sequence $x = \{\xi_k\}$ is said to be strongly almost (V,λ)-summable to a number ξ if

$$\lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \qquad \text{uniformly in m.}$$

In this case we write $\xi_k \to \xi[\hat{V}, \lambda]$ and $[\hat{V}, \lambda]$ denotes the set of all strongly almost (V,λ)-summable sequences,

i.e. $$[\hat{V}, \lambda] = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{n\to\infty} \frac{1}{\lambda_n} \sum_{k \in I_n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m}\}.$$

Definition 1.1.11. A sequence $x = \{\xi_k\}$ is said to be almost statistically convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{k \le n: |\xi_{k+m} - \xi| \ge \varepsilon\}| = 0 \quad \text{uniformly in m.}$$

In this case we write $\hat{S}$-lim $\xi_k = \xi$ or $\xi_k \to \xi(\hat{S})$ and $\hat{S}$ denotes the set of all almost statistically convergent sequences.

Definition 1.1.12. A sequence $x = \{\xi_k\}$ is said to be almost λ-statistically convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \ge \varepsilon\}| = 0 \text{ uniformly in m.}$$

In this case we write $\hat{S}_\lambda$-lim $\xi_k = \xi$ or $\xi_k \to \xi(\hat{S}_\lambda)$ and $\hat{S}_\lambda$ denotes the set of all almost λ-statistically convergent sequences.

Remark 1.1.13. If $\lambda_n = n$, then $\hat{S}_\lambda$ is same as $\hat{S}$.

1.2 SOME INCLUSION RELATION BETWEEN ALMOST λ-STATISTICAL CONVERGENCE, STRONG ALMOST (V,λ)-SUMMABILITY AND STRONG ALMOST CONVERGENCE

In this section we study some inclusion relations between almost λ-statistical convergence, strong almost (V,λ)-summability and strong almost convergence. First we show that every strongly almost summable sequence is almost statistically convergent.

Theorem 1.4.1. If a sequence $x = \{\xi_k\}$ is almost strongly summable to ξ, then it is almost statistically convergent to ξ.

Proof. Suppose that $x = \{\xi_k\}$ is almost strongly summable to ξ. Then

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m.} \quad \ldots(1)$$

Let us take some ε > 0. We have

$$\sum_{k=1}^{n} |\xi_{k+m} - \xi| \geq \sum_{|\xi_{k+m}-\xi| \geq \varepsilon} |\xi_{k+m} - \xi|$$

$$\geq \varepsilon |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by (1) and the fact that ε is fixed number, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \text{ uniformly in m.}$$

⇒ x is almost statistically convergent.

Theorem 1.2.2. Let $\lambda = \{\lambda_n\}$ be same as defined earlier. Then

(i) $\xi_k \to \xi[\hat{V},\lambda] \Rightarrow \xi_k \to \xi(\hat{S}_\lambda)$

and the inclusion $[\hat{V},\lambda] \subseteq \hat{S}_\lambda$ is proper,

(ii) if $x \in l_\infty$ and $\xi_k \to \xi(\hat{S}_\lambda)$, then $\xi_k \to \xi[\hat{V},\lambda]$ and hence $\xi_k \to \xi[\hat{c}]$ provided $x = \{\xi_k\}$ is not eventually constant.

(iii) $\hat{S}_\lambda \cap l_\infty = [\hat{V},\lambda] \cap l_\infty$,

where l_∞ denotes the set of bounded sequences.

Proof. (i). Since $\xi_k \to \xi[\hat{V},\lambda]$, for each ε > 0, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = 0 \qquad \text{uniformly in m.} \quad \ldots(2)$$

Let us take some ε > 0. We have

$$\sum_{k\in I_n} |\xi_{k+m} - \xi| \geq \sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi| \geq \varepsilon}} |\xi_{k+m} - \xi|$$

$$\geq \varepsilon |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Consequently,

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| \geq \varepsilon \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Hence by using (2) and the fact that ε is fixed number, we have

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \qquad \text{uniformly in m,}$$

i.e. $\xi_k \to \xi(\hat{S}_\lambda)$.

It is easy to see that $[\hat{V},\lambda] \subset \hat{S}_\lambda$.

(ii). Suppose that $\xi_k \to \xi(\hat{S}_\lambda)$ and x $\in l_\infty$. Then for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \quad \text{uniformly in m.} \quad \ldots(3)$$

Since x $\in l_\infty$, there exists a positive real number M such that $|\xi_{k+m} - \xi| \leq M$ for all k and m. For given $\varepsilon > 0$, we have

$$\frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi|\geq\varepsilon}} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi|<\varepsilon}} |\xi_{k+m} - \xi|$$

$$\leq \frac{1}{\lambda_n}\sum_{\substack{k\in I_n \\ |\xi_{k+m}-\xi|\geq\varepsilon}} M + \frac{1}{\lambda_n}\sum_{k\in I_n} \varepsilon$$

$$= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} [n-(n-\lambda_n + 1) + 1]$$

$$= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{\lambda_n} \lambda_n$$

$$= \frac{M}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon$$

$$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| \leq M \lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| + \varepsilon$$

Hence by using (3), we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi| = 0 \quad \text{uniformly in m.} \quad \ldots(4)$$

$$\Rightarrow \quad \xi_k \to \xi[\hat{V},\lambda].$$

Further, we have

$$\frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = \frac{1}{n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n}\sum_{k=n-\lambda_n+1}^{n} |\xi_{k+m} - \xi|$$

$$= \frac{1}{n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$$\leq \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} |\xi_{k+m} - \xi| + \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$$\leq \frac{2}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$$

$\Rightarrow \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| \leq 2 \lim_{n\to\infty} \frac{1}{\lambda_n}\sum_{k\in I_n} |\xi_{k+m} - \xi|$

Hence

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} |\xi_{k+m} - \xi| = 0 \text{ uniformly in m. [Using (4)]}$$

$$\Rightarrow \quad \xi_k \to \xi[\hat{c}].$$

(iii). Let $x \in l_\infty$ be such that $\xi_k \to \xi\, (\hat{S}_\lambda)$.

Then by (ii),

$$\xi_k \to \xi[\hat{V},\lambda].$$

Thus

$$\hat{S}_\lambda \cap l_\infty \subset [\hat{V},\lambda] \cap l_\infty. \qquad \ldots(5)$$

Also by (i), we have

$\xi_k \to \xi[\hat{V},\lambda] \Rightarrow \xi_k \to \xi(\hat{S}_\lambda)$.

So $\quad [\hat{V},\lambda] \subset \hat{S}_\lambda$.

$$\Rightarrow \quad [\hat{V},\lambda] \cap l_\infty \subset \hat{S}_\lambda \cap l_\infty. \qquad \ldots(6)$$

Hence by (5) and (6)

$$\hat{S}_\lambda \cap l_\infty = [\hat{V},\lambda] \cap l_\infty.$$

This completes the proof of the theorem

1.3 NECESSARY AND SUFFICIENT CONDITION FOR AN ALMOST STATISTICALLY CONVERGENT SEQUENCE TO BE ALMOST λ-STATISTICALLY CONVERGENT

Since $\frac{\lambda_n}{n}$ is bounded by 1, we have $\hat{S}_\lambda \subseteq \hat{S}$ for all λ. In this section we discuss the following relation.

Theorem 1.4.1. $\hat{S} \subseteq \hat{S}_\lambda$ if and only if

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} > 0, \qquad \ldots(7)$$

i.e. every almost statistically convergent sequence is almost λ-statistically convergent if and only if (7) holds.

Proof. Let us take an almost statistically convergent sequence $x = \{\xi_k\}$ and assume that (7) holds.

Then for each $\varepsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{n} |\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \text{ uniformly in m.} \qquad \ldots(8)$$

For given $\varepsilon > 0$ we get,

$$\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\} \supset \{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}.$$

Therefore,

$$\frac{1}{n}|\{k \leq n: |\xi_{k+m} - \xi| \geq \varepsilon\}| \geq \frac{1}{n}|\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

$$\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n: |\xi_{k+m} - \xi| \geq \varepsilon\}|$$

Taking the limit as $n\to\infty$ and using (7), we get

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{k \in I_n : |\xi_{k+m} - \xi| \geq \varepsilon\}| = 0 \qquad \text{uniformly in } m,$$

i.e. $\xi_k \to \xi(\hat{S}_\lambda)$.

Hence $\hat{S} \subseteq \hat{S}_\lambda$ for all λ.

Conversely, suppose that $\hat{S} \subseteq \hat{S}_\lambda$ for all λ.

We have to prove that (7) holds.

Let as assume that

$$\liminf_{n\to\infty} \frac{\lambda_n}{n} = 0.$$

As in [9], we can choose a subsequence $\{n(j)\}$ such that

$$\frac{\lambda_{n(j)}}{n(j)} < \frac{1}{j}.$$

Define a sequence $x = \{\xi_i\}$ by

$$\xi_i = \begin{cases} 1 & \text{if } i \in I_{n(j)}, j = 1,2,3,\ldots \\ 0 & \text{otherwise.} \end{cases}$$

Then $x \in [\hat{c}]$ and hence by Theorem 1.4.1, $x \in \hat{S}$. But on the other hand $x \notin [\hat{V},\lambda]$ and Theorem 1.4.1 (ii) implies that $x \notin \hat{S}_\lambda$. Hence (7) is necessary.

This completes the proof of the theorem

References:

[1] I.J. Maddox, Statistical convergence in a locally convex space, *Math. Proc. Camb. Phil. Soc.* 104 (1988) 141-145.

[2] Mursaleen, Invariant means and some matrix transformations, *Tamkang J. Math.* 10 (2) (1979) 183-188.

[3] Mursaleen, Matrix transformations between some new sequence spaces, *Houston J. Math.* 9 (4) (1983) 505-509.

[4] Mursaleen, λ-statistical convergence, *Math. Slovaca* 50 (1) (2000) 111-115.

[5] I. Niven and H.S. Zuckerman, *An Introduction to The Theory of Numbers*, Fourth Ed., New York, John Wiley and Sons, 1980.

[6] T. Šalát, On statistically convergent sequences of real numbers, *Math. Slovaca* 30 (2) (1980) 139-150.

[7] E. Savas, Some sequence spaces involving invariant means, *Indian J. Math.* 31 (1) (1989) 1-8.

[8] E. Savas, On lacunary strong σ-convergnce, *Indian J. Pure Appl. Math.* 21 (4) (1990) 359-365.

[9] E. Savas and F. Nuray, On σ-statistical convergence and lacunary σ-statistical convergence, *Math. Slovaca* 43 (1993) 309-315.

[10] E. Savas, Strong almost convergence and almost λ-statistical convergence, *Hokkaido Math. J.* 29 (2000) 531-536.

[11] P. Schaefer, Infinite matrices and invariant means, *Proc. Amer. Math. Soc.* 36 (1) (1972) 104-110.

[12] I.J. Schoenberg, The integrability of certain functions and related summability methods, *Amer. Math. Monthly* 66 (1959) 361-375.

9/30/2021