



On Jost Solutions Of Sturm-Liouville Equations With Spectral Parameter In Discontinuity Condition

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Abstract:

Integral representation

Consider the differential equation

$$-y'' + q(x)y = \lambda^2 y, x \in (-\infty, -\infty) \quad (1)$$

with the conditions

$$y(a+0) = y(a-0),$$

$$y'(a+0) = y'(a-0) = \lambda \beta y(a), \quad (3)$$

where $\beta, a \in (-\infty, +\infty), \beta \neq 0, \lambda$ is a complex parameter, $q(x)$ is a real-valued function and satisfies the condition

$$\int_{-\infty}^{+\infty} (1 + |x|)|q(x)|dx < +\infty. \quad (4)$$

We can understand problem(1) – (3) as one of the treatments of the equation

$$-y'' + \lambda p(x)y + q(x)y = \lambda^2 y, -\infty < x < +\infty, \quad (5)$$

$$\text{When } p(x) = \beta \delta(x - a).$$

When the function $p(x)$ is sufficiently smooth, the functions $p(x)$ and $q(x)$ are real-valued and decrease quite rapidly, the inverse scattering problem for equation (5) is completely solved in the papers [1] – [4].

In order to solve such a problem in the case $p(x) = \beta \delta(x - a)$, in the present report we prove the existence of Jost type solutions for problem(1) – (3) and investigate their properties.

The functions $e^\pm(x, \lambda)$ satisfying equation (1), conditions (1) – (3) and the condition at infinity

$$\lim_{x \rightarrow \pm\infty} e^\pm(x, \lambda) \cdot e^{\mp i\lambda x} = 1, \quad (6\pm)$$

are called the Jost solutions.

It is easy to show that if $q(x) \equiv 0$, the functions

$$e_0^\pm(x, \lambda) = \begin{cases} e^{\pm i\lambda x}, & \pm x > \pm a, \\ \left(1 + \frac{i\beta}{2}\right) e^{\pm i\lambda x} - \frac{i\beta}{2} e^{\pm i\lambda(2a-x)}, & \pm x < \pm a. \end{cases}$$

are the Jost solutions.

Introduce the following denotation:

$$c = 1 + |\beta|, \sigma_1^\pm(x) = \pm \int_x^{\pm\infty} t|q(t)|dt.$$

The basic result of the paper is the following

Theorem: Let the real-valued function $q(x)$ satisfy condition (4). There for all λ , there exist Jost solutions $e^\pm(x, \lambda)$ of (1) – (3) from the upper half-plane, they are unique and represented in the form

$$e^\pm(x, \lambda) = e_0^\pm(x, \lambda) \pm \int_x^{\pm\infty} K^\pm(x, t) e^{\pm i\lambda t} dt, \quad (7\pm)$$

where for each fixed $x \neq a$ the kernel $K^+(x, \cdot)$ ($K^-(x, \cdot)$) belongs to the space $L_1(x, +\infty)$ ($L_1(-\infty, x)$) and the estimations



$$\pm \int_x^{\pm\infty} |K^+(x, t)| dt \leq e^{c\sigma_{\pm}^{\pm}(x)} - 1. \quad (8\pm)$$

Are fulfilled. Futhermore

$$\begin{aligned} K^{\pm}(x, x) &= \pm \frac{1}{2} \int_x^{\pm\infty} q(\xi) d\xi, \quad \pm x > \pm a, \\ K^{\pm}(x, x) &= \pm \frac{1}{2} \left(1 + \frac{i\beta}{2}\right) \int_x^{\pm\infty} q(\xi) d\xi, \quad \pm x < \pm a, \\ K^{\pm}(x, 2a - x + 0) - K^{\pm}(x, 2a - x - 0) &= \\ &= \frac{i\beta}{4} \left\{ \int_x^a q(\xi) d\xi - \int_a^{+\infty} q(\xi) d\xi \right\}, \quad \pm x < \pm a. \end{aligned} \quad (9\pm)$$

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1. Introduction

Remark. When no discontinuity condition exists, i. e. when in condtion (3) $\beta = 0$, the representation of the Jost solution for Sturm – liouville equation is first obtained in the paper [5](see also[6]). Such problem for the equation $-\mathcal{Y}'' + q(x)\mathcal{Y} = \lambda p(x)\mathcal{Y}$, when $p(x)$ is a piecewise – constant real function is solved in the paper[7].

Theorems proof. Having rewritten equation (1) in the from

$\mathcal{Y}'' + \lambda^2 \mathcal{Y} = q(x)\mathcal{Y}$ and assuming the right hand side be known, for funding the solution $e^+(x, \lambda)$ of this equation we can apply the arbitrary constants variation method As a result, we get the integral equation

$$e^+(x, \lambda) = e_0^+(x, \lambda) + \int_x^{+\infty} S_0(t, x, \lambda) q(t) e^+(t, \lambda) dt, \quad (10)$$

Where

$$S_0(t, x, \lambda) = \begin{cases} \frac{\sin\lambda(t-x)}{\lambda}, & t > x > a \text{ or } a > t > x, \\ \frac{\sin\lambda(t-x)}{\lambda} - \frac{\beta}{2} \frac{\cos\lambda(t-x) - \cos\lambda(t-2a+x)}{\lambda}, & t > a > x \end{cases} \quad (11)$$

It is easy to show that the solution $e^+(x, \lambda)$ of integral equation(10) is the Jost solution of problem (1) – (3), (6₊). Well look for the solution of equation (10) in the form (7₊) In order such kind function satisfy equation(10), the equality.

$$\begin{aligned} \int_x^{+\infty} K^+(x, t) e^{i\lambda t} dt &= \int_x^{+\infty} S_0(t, x, \lambda) q(t) e_0^+(t, \lambda) dt + \\ &+ \int_x^{+\infty} S_0(t, x, \lambda) q(t) \int_t^{+\infty} K^+(t, s) e^{i\lambda s} ds dt, \end{aligned} \quad (12)$$

should be fulfilled. And vice – versa, if $K^+(x, t)$ satisfies this equality for all $\lambda (\text{Im}\lambda \geq 0)$,

The function $e^+(x, \lambda)$ is the Jost solution of problem (1) – (3), (6₊).

Transform each term in the right side of (12) so that they have the form of the Fourier transform of some functions.

At first we consider the first term. It $x < a$, we have

$$\begin{aligned} & \int_x^{+\infty} S_0(t, x, \lambda)q(t)e_0^+(t, \lambda)dt = \\ &= \int_x^a \frac{\sin\lambda(t-x)}{\lambda} q(t) \left[\left(1 + \frac{i\beta}{2}\right) e^{i\lambda t} - \frac{i\beta}{2} e^{i\lambda(2a-t)} \right] dt + \\ &+ \int_a^{+\infty} \left[\frac{\sin\lambda(t-x)}{\lambda} - \frac{\beta}{2} \cdot \frac{\cos\lambda(t-x) - \cos\lambda(t-2a+x)}{\lambda} \right] q(t) e^{i\lambda t} dt = \\ &= \left(1 + \frac{i\beta}{2}\right) \int_x^a \left(\frac{1}{2} \int_x^{2t-x} e^{i\lambda\xi} d\xi\right) q(t) dt - \frac{i\beta}{2} \int_x^a \left(\frac{1}{2} \int_{x-2t+2a}^{2a-x} e^{i\lambda\xi} d\xi\right) q(t) dt + \\ &+ \int_a^{+\infty} \int_x^a \left(\frac{1}{2} \int_x^{2t-x} e^{i\lambda\xi} d\xi\right) q(t) dt - \int_a^{+\infty} \left(\frac{1}{2} \int_{2t-2a+x}^{2t-x} e^{i\lambda\xi} d\xi\right) q(t) dt + \\ &+ \frac{i\beta}{2} \int_a^{+\infty} \left(\frac{1}{2} \int_x^{2a-x} e^{i\lambda\xi} d\xi\right) q(t) dt. \end{aligned}$$

Changing the integration order and then in the obtained equality changing the denotation for integration variables, we get (for $x < a$)

$$\begin{aligned} & \int_x^{+\infty} S_0(t, x, \lambda)q(t)e_0^+(t, \lambda)dt = \frac{1}{2} \left(1 + \frac{i\beta}{2}\right) \int_x^{2a-x} \left(\int_{\frac{t+x}{2}}^a q(\xi) d\xi\right) e^{i\lambda t} dt - \\ &- \frac{i\beta}{4} \int_x^{2a-x} \left(\int_{\frac{x+2a-t}{2}}^a q(\xi) d\xi\right) e^{i\lambda t} dt + \frac{1}{2} \int_x^{2a-x} \left(\int_a^{+\infty} q(\xi) d\xi\right) e^{i\lambda t} dt + \\ &+ \frac{1}{2} \int_{2a-x}^{+\infty} \left(\int_{\frac{t+x}{2}}^{+\infty} q(\xi) d\xi\right) e^{i\lambda t} dt - \frac{i\beta}{4} \int_x^{2a-x} \left(\int_a^{\frac{t+2a-x}{2}} q(\xi) d\xi\right) e^{i\lambda t} dt - \\ &- \frac{i\beta}{4} \int_{2a-x}^{+\infty} \left(\int_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}} q(\xi) d\xi\right) e^{i\lambda t} dt + \frac{i\beta}{4} \int_x^{2a-x} \left(\int_a^{+\infty} q(\xi) d\xi\right) e^{i\lambda t} dt = \\ &= \frac{1}{2} \int_x^{+\infty} \left(\int_{\frac{t+x}{2}}^{+\infty} e^{i\lambda\xi} d\xi\right) e^{i\lambda t} dt + \frac{i\beta}{4} \int_x^{2a-x} \left(\int_{\frac{t+x}{2}}^{+\infty} q(\xi) d\xi - \int_{\frac{x+2a-t}{2}}^{\frac{t+2a-x}{2}} q(\xi) d\xi\right) e^{i\lambda t} dt - \\ &- \frac{i\beta}{4} \int_{2a-x}^{+\infty} \left(\int_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}} q(\xi) d\xi\right) e^{i\lambda t} dt. \end{aligned} \tag{13}$$

For $x > a$ we behave in the similar way and have

$$\begin{aligned} & \int_x^{+\infty} S_0(t, x, \lambda)q(t)e_0^+(t, \lambda)dt = \int_x^{+\infty} \frac{\sin\lambda(t-x)}{\lambda} q(t) e^{i\lambda t} dt = \\ &= \int_x^{+\infty} \left(\frac{1}{2} \int_x^{2t-x} e^{i\lambda\xi} d\xi\right) q(t) dt = \frac{1}{2} \int_x^{+\infty} \left(\int_{\frac{x+t}{2}}^{+\infty} q(\xi) d\xi\right) e^{i\lambda t} dt. \end{aligned} \tag{14}$$

Now, transform the second term form the right hand side of relation(12). For $x < a$ we have

$$\begin{aligned} & \int_x^{+\infty} S_0(\xi, x, \lambda)q(\xi) \int_{\xi}^{+\infty} K^+(x, u) e^{i\lambda u} du ds = \\ &= \int_x^{+\infty} \frac{\sin\lambda(\xi-x)}{\lambda} q(\xi) \int_{\xi}^{+\infty} K^+(\xi, u) e^{i\lambda u} dud\xi - \\ &- \frac{\beta}{2} \int_a^{+\infty} \frac{\cos\lambda(\xi-x) - \cos\lambda(\xi-2a+x)}{\lambda} q(\xi) \int_{\xi}^{+\infty} K^+(\xi, u) e^{i\lambda u} dud\xi = \\ &= \frac{1}{2} \int_x^{+\infty} q(\xi) \int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{x-\xi+u}^{\xi-x+u} e^{i\lambda t} dt \right\} dud\xi - \end{aligned}$$

$$-\frac{i\beta}{4} \int_a^{+\infty} q(\xi) \int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t} dt \right\} dud\xi +$$

$$+\frac{i\beta}{4} \int_a^{+\infty} q(\xi) \int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t} dt \right\} dud\xi.$$

Continuing the function $K^+(\xi, u)$ by a zero for $u < \xi$, for all $t \geq x$ we find

$$\int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{x+u-\xi}^{\xi-x+u} e^{i\lambda t} dt \right\} du = \int_{-\infty}^{+\infty} K^+(\xi, u) \left\{ \int_{x+u-\xi}^{\xi-x+u} e^{i\lambda t} dt \right\} du$$

$$= \int_{-\infty}^{+\infty} \left(\int_{t-\xi+x}^{t+\xi-x} K^+(\xi, u) du \right) e^{i\lambda \xi} d\xi = \int_x^{\infty} \left(\int_{t-\xi+x}^{t+\xi-x} K^+(\xi, u) du \right) e^{i\lambda t} dt, \quad (15)$$

Since for $t < x$

$$\int_{t-\xi+x}^{t+\xi-x} K^+(\xi, u) du = 0.$$

Behaving in the same way, for all $t \geq a$ we have

$$\int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t} dt \right\} du = \int_{-\infty}^{+\infty} \left\{ \int_{\xi-2a+x+u}^{\xi-x+u} e^{i\lambda t} dt \right\} K^+(\xi, u) du =$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi, u) du \right\} e^{i\lambda t} dt = \int_x^{+\infty} \left\{ \int_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi, u) du \right\} e^{i\lambda t} dt, \quad (16)$$

$$\int_{\xi}^{+\infty} K^+(\xi, u) \left\{ \int_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t} dt \right\} du = \int_{-\infty}^{+\infty} \left\{ \int_{x+u-\xi}^{2a-x+u-\xi} e^{i\lambda t} dt \right\} K^+(\xi, u) du =$$

$$= \int_{-\infty}^{+\infty} \left\{ \int_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi, u) du \right\} e^{i\lambda t} dt = \int_x^{+\infty} \left\{ \int_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi, u) du \right\} e^{i\lambda t} dt, \quad (17)$$

Here, we used the fact that for $t < x$

$$\int_{x+t-\xi}^{t+2a-x-\xi} K^+(\xi, u) du = 0, \quad \int_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi, u) du = 0.$$

It follows from formulae (13) – (17) that equality (12) is fulfilled if the function $K^+(x, t)$ satisfies the equation

$$K^+(x, t) = K_0^+(x, t) + \frac{1}{2} \int_x^{+\infty} q(\xi) \int_{t-\xi+x}^{t+\xi-x} K^+(\xi, u) dud\xi -$$

$$-\frac{i\beta}{4} H^+(x) \int_a^{+\infty} q(\xi) \int_{t-\xi+x}^{t-\xi+2a-x} K^+(\xi, u) dud\xi +$$

$$-\frac{i\beta}{4} H^+(x) \int_a^{+\infty} q(\xi) \int_{t-2a+x+\xi}^{t-x+\xi} K^+(\xi, u) dud\xi, \quad (18)$$

Where

$$H^+(x) = \begin{cases} 1, & x < a, \\ 0, & x > a, \end{cases}$$

$$K_0^+(x, t) = \frac{1}{2} \int_{\frac{t+x}{2}}^{+\infty} q(s) ds - \frac{i\beta}{4} H(x) \times$$

$$\times \begin{cases} \int_{\frac{x+2a-t}{2}}^{\frac{t+2a-x}{2}} q(s) ds - \int_{\frac{x+t}{2}}^{+\infty} q(s) ds, & x < t < 2a - x, \\ \int_{\frac{t+x}{2}}^{\frac{t+2a-x}{2}} q(s) ds, & 2a - x < t < \infty. \end{cases} \quad (19)$$

Thus, in order to complete the proof of existence of the solution $e^+(x, \lambda)$ it is enough to show that for each fixed $x \in (-\infty, a) \cup (a, +\infty)$ equation (18) has the solution $K^+(x, 0) \in L_1(x, +\infty)$ satisfying inequality (8₊) and condition (9₊). Assume

$$K_n^+(x, t) = \frac{1}{2} \int_x^{+\infty} q(\xi) \int_{t-\xi+x}^{t+\xi-x} K_{n-1}^+(\xi, u) du d\xi - \frac{i\beta}{4} H(x) \left\{ \int_a^{+\infty} q(\xi) \int_{t-\xi+x}^{t-\xi+2a-x} K_{n-1}^+(\xi, u) du d\xi - \int_a^{+\infty} q(\xi) \int_{t-2a+\xi+x}^{t-x+\xi} K_{n-1}^+(\xi, u) du d\xi \right\}, n = 1, 2, \dots, \quad (20)$$

Where $K_0^+(x, t)$ is determined by formula (19).

$$\int_x^{+\infty} |K_n^+(x, t)| dt \leq \frac{c^{n+1} \sigma_1^{n+1}(x)}{(n+1)!} \quad (21)$$

Whence, it will follow that the series $K^+(x, 0) = \sum_{n=0}^{+\infty} K_n^+(x, \cdot)$ converges in

the space $L_1(x, +\infty)$, its sum $K^+(x, t)$ is a solution of integral equation (18) and satisfies estimation (8₊).

It follows from the definition of $K_n^+(x, t)$ (see formula 20) that

$$|K_n^+(x, t)| \leq c \int_x^{+\infty} |q(\xi)| \int_{t-\xi+x}^{t+\xi-x} |K_{n-1}^+(\xi, u)| du d\xi,$$

Consequently,

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$$\int_x^{+\infty} |K_n^+(x, t)| dt \leq c \int_x^{+\infty} \xi |q(\xi)| \int_{\xi}^{+\infty} |K_{n-1}^+(\xi, u)| du d\xi. \quad (22)$$

Now, for establishing inequality (21), apply the mathematical induction method.

For $n = 0$ use (19), change the integration order and have

$$\begin{aligned} \int_x^{+\infty} |K_0^+(x, t)| dt &\leq \int_x^{+\infty} (\xi - x) |q(\xi)| d\xi + \frac{|\beta|}{2} H(x) \int_x^a (\xi - x) |q(\xi)| d\xi + \\ &+ \int_a^{2a-x} (2a - x - \xi) |q(\xi)| d\xi + \int_x^a (\xi - x) |q(\xi)| d\xi + \int_a^{+\infty} (a - x) |q(\xi)| d\xi + \\ &+ \int_a^{2a-x} (\xi - a) |q(\xi)| d\xi + \int_{2a-x}^{+\infty} (a - x) |q(\xi)| d\xi = \int_x^{+\infty} (\xi - x) |q(\xi)| d\xi + \\ &+ \beta H(x) \left\{ \int_x^a (\xi - a) |q(\xi)| d\xi + \int_a^{+\infty} (a - x) |q(\xi)| d\xi \right\} \leq c \int_x^{+\infty} \xi |q(\xi)| d\xi = c \sigma_1(x). \end{aligned}$$

Thus, estimation (21) is true for $n = 2$ and if it true for $\|K_n^+(x, \cdot)\|_{L_1(x, +\infty)}$, then

Using inequality (22), we have

$$\int_x^{+\infty} |K_{n+1}^+(x, t)| dt \leq c \int_x^{+\infty} \xi |q(\xi)| \frac{c^{n+1} \sigma_1^{n+1}(\xi)}{(n+1)!} d\xi = \frac{c^{n+2} \sigma_1^{n+2}(\xi)}{(n+2)!}$$

Validity of relations (9) follows directly from (18) – (19) Form the integral Equation

$$e^-(x, \lambda) = e_0^-(x, \lambda) + \int_{-\infty}^x S_0(x, t, \lambda) q(t) e^-(t, \lambda) dt,$$

the proof of the theorem statement related with the solution $e^-(t, \lambda)$ is carried out

In the similar way, Here we notice only the integral equation for the kernel

$$\begin{aligned} K^-(x, t) &= K_0^-(x, t) + \frac{1}{2} \int_{-\infty}^x q(\xi) \int_{t-x+\xi}^{t-\xi+x} K^-(\xi, u) du d\xi - \\ &- \frac{i\beta}{4} H^-(x) \left\{ \int_{-\infty}^a q(\xi) \int_{t-2a+x+\xi}^{t-\xi+x} K^-(\xi, u) du d\xi - \right. \\ &\left. \int_{-\infty}^a q(\xi) \int_{t-x+\xi}^{t-x+2a-\xi} K^-(\xi, u) du d\xi \right\} \end{aligned}$$

where

$$K_0^-(x, t) + \frac{1}{2} \int_{-\infty}^{\frac{t+x}{2}} q(\xi) d\xi - \frac{i\beta}{4} H^-(x) \begin{cases} \int_{\frac{t-x+2a}{2}}^{\frac{t+x}{2}} q(\xi) d\xi, & -\infty < t < 2a, -x \\ \int_{\frac{t-x+2a}{2}}^{\frac{x+2a-t}{2}} q(\xi) d\xi - \int_{-\infty}^{\frac{x+t}{2}} q(\xi) d\xi, & 2a - x < t < x. \end{cases}$$

$$H^-(x) = \begin{cases} 1, & x > a, \\ 0, & x < a. \end{cases}$$

The theorem is proved.

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References

- [1]. Jaulent . M. On an ineverse scattering problem with an energy – depe –ndet potential . Ann Inst . Henri Paunkare ,1972, v. 7. No 4 , pp. 363 – 378.
- [2]. Jaulent M. On the ineverse problem for the schrodinger equation with an Energy – dependet potential . Comptes rendus Akad . Sci. Paris , 1975 , v. : 280 , serie A. , pp. 1467 – 1470.
- [3]. Jaulent M. , Jean C. Inverse problem for the one dimensional schrodinger Equation with an energy – degendet potential . Ann Inst. Henri Paunkare, 1976, v. 25, No 2, pp. 105 – 137.
- [4]. Maksudov F. G. , Huseynov G. Sh . To the solution of the inverse scattering Problem for a quadratic bundle of Schrodinger one – dimensional operators on the axis. DAN SSSR ,1486 , v. 289, No 1, pp. 42 – 46(Russian).
- [5]. Levin B. Ya Fourier and Laplace type transformation by means of the solutions of a second order differential equation . DAN SSSR , 1956 , vol .106, No 2, pp. 187 –190(Russian).
- [6]. Marchenko V. A. Sturm – Liouville operators and their applications . Kiev, Naukova Dumka ,1977 , p331(Russian).
- [7]Huseynov I. M. Pashaev R. T. On an inverse problem for a second order different equation . Uspkhi matem , Nauk , 20, issue 7, pp. 143 – 148(Russian).