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Review Of Literature Related To Operational Research In Study Of Mathematics

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Abstract: Some pattern-forming mechanisms have an intrinsic tendency to form stripes. This involves competition between emerging patterns which remains intuitively not obvious to us although we can establish it by computation and, in part, by analysis we describe here. Our conclusions are applicable to a wide range of pattern-forming mechanisms within the general category of kinetic mechanisms. These include reaction diffusion models, first proposed by Turing (1952) and to date the most extensively developed kind of kinetic mechanism. The category also includes mechanochemical theories (Oster et al., 1983) as well as mechanisms involving complex cell-cell interactions, for example between groups of incipient synapses in the assembly of the nervous system. These last are involved in the formation of ocular dominance patterns in the primary visual cortex (also known as the striate cortex, area 17 or V1) of higher vertebrates (Fig. 1B). In an earlier publication (Lyons and Harrison, 1991) we showed that the observed patterns are similar to those which can be modelled using reaction-diffusion systems. The present analysis and discussion is applicable to these disparate mechanisms.
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Introduction:

Various studies (e.g. Stacey, 1988; Vinner, 1991; Kieran, 1992; Esty, 1992; Sfard & Linchevski, 1994; Bell, 1995; Linchevski & Herscovics, 1996; McDowell, 1996; Souviney, 1996; Dreyfus, 1999; Lithner, 2000; Mason, 2000, Maharaj, 2005) have focused on the teaching and learning of school mathematics. These studies have indicated some important sources of students' difficulties in mathematics. Kieran (1992) considered a student's inability to acquire an in-depth sense of the structural aspects of algebra to be the main obstacle. Sfard and Linchevski (1994) have analysed the nature and growth of algebraic thinking from an epistemological perspective supported by historical observations. They indicated that the development of algebraic thinking was a sequence of ever more advanced transitions from operational (procedural) to structural outlooks. Mason (2000:97) has argued that "... the style and the nature of questions encountered by students strongly influences the sense that they make of the subject matter". The questions that come to the mind of an educator are influenced by the perspective and disposition that he/she has towards mathematics and pedagogy (Mason, 2000).

These questions in turn influence the sense learners make of the subject matter. In this article I focus on the outcomes and implications of research on (a) use of symbols in mathematics, (b) algebraic/trigonometric expressions, (c) solving equations, and (d) functions and calculus.

In seeking to explain the complex phenomena of biological pattern formation, one must start with an a priori concept of where the complexity lies. Wortis et al. (in press) have drawn the contrast between complex machines and simple machines with complex behaviour. Molecular biologists seek the former: the complex machine as a multiplicity of genes and gene products, mutually governed by many regulative processes which are complex by their sheer number but each rather simple in character. Physical scientists tend to seek the latter: dynamic processes which can be described by a few simple terms in two or three equations, but which display complex behaviour.

This research describes work within the latter paradigm. Our results rely on mathematical analysis to an extent; however, for the study of wide ranges of parameter values, our method has been, as in most work of this kind, to put the model into a computer and take its workings as a topic for experimental study in much the 0 1993 WILEY-LISS, INC. same spirit as an experimental biologist's study of a developmental phenomenon. Such studies are undertaken as contributions to the theory of natural phenomena, in this case biological pattern formation. Nevertheless, while it is in progress, the work resembles an experimental study in its own right, with the model as the subject. An important class of biological phenomena is the generation of two-dimensional structures periodically repeated in space. Such patterns occur in diverse biological contexts, for example (1) a multicellular epithelial sheet, as for the striped and spotted coat patterns of many mammals; (2) a multinucleate syncytium having a single layer of some thousands of nuclei, such as the Drosophila blastoderm in which striped patterns of pair-rule gene-expression arise; (3) the surface of a large single cell, for instance the growing tip of the alga Acetabularia which generates both vegetative and reproductive whorls at a site which may be centimeters from the only nucleus (Harrison et al., 1988; Berger et al., 1987).

The lobes of the alga Micrasterias (Lacalli and Harrison, 1987) and some microtubule arrays near the cell surfaces of ciliates such as Paramecium and Tetrahymena (Frankel, 1989) are further unicellular examples. Such phenomena show the ability of a living organism to establish a quantitative measure of spacing between adjacent repeats of similar structures and to use it repeatedly in the same direction. Here, "repeatedly" is not intended to imply a time sequence in which structures are formed one by one. In some cases, very numerous parts of the overall pattern are expressed precisely simultaneously, e.g., up to 80 striations in the reproductive cap primordium of Acetabularia. In a large taxonomic group wherein one may expect the mechanism for formation of a particular kind of pattern to have been conserved, the pattern formation process can be essentially simultaneous in some species and time sequential in others (e.g., the onset of seg mentation in the class Insecta). In this research, we consider the problem of generating the parts of a pattern simultaneously. This approach does not lack generality.

Review of literature:

The cognitive load work by Kirschner, Sweller, and Clark (2006) gives an explanation for the necessity of fluency with prerequisite knowledge. Without prerequisite fluency, short-term memory becomes overloaded and unable to effectively process the new concepts being learned.

Boyer's (1968) A History of Mathematics is almost entirely about Greek mathematics. It covers ancient Greek mathematics to a degree that none of the other mentioned texts do. Perhaps one of the most valuable tools for a secondary teacher available is Historical Topics for the Mathematics Classroom (National Council for Teachers of Mathematics, 1989).

This text consists of a series of "capsules" (short chapters). Each capsule gives a brief historical overview of a particular topic (e.g. Napier's Rods). The capsules are grouped by general topic (algebra, geometry, trigonometry, etc.). Specifically, this text provides a historical context to graphical approaches to equation solving. In addition, it provides a concise overview of the methods employed to solve quadratics and cubics.

Various researchers (Vaiyavutjamai & Clements, 2006) have illustrated that very little attention has been paid to quadratic equations in mathematics education literature, and there is scarce research regarding the teaching and learning of quadratic equations.

A limited number of research studies focusing on quadratic equations have documented the techniques students engage in while solving quadratic equations (Bossé & Nandakumar, 2005), geometric approaches used by students for solving quadratic equations (Allaire & Bradley, 2001), students' understanding of and difficulties with solving quadratic equations (Kotsopoulos, 2007; Lima, 2008; Tall, Lima, & Healy, 2014; Vaiyavutjamai, Ellerton, & Clements, 2005; Zakaria & Maat, 2010), the teaching and learning of quadratic equations in classrooms (Olteanu & Holmqvist, 2012; Vaiyavutjamai & Clements, 2006), comparing how quadratic equations are handled in mathematics textbooks in different countries (Saglam & Alacaci, 2012), and the application of the history of quadratic equations in teacher preparation programs to highlight prospective teachers' knowledge (Clark, 2012).

In general, for most students, quadratic equations create challenges in various ways such as difficulties in algebraic procedures, (particularly in factoring quadratic equations), and an inability to apply meaning to the quadratics. Kotsopoulos (2007) suggests that recalling main multiplication facts directly influences a student's ability while engaged in factoring quadratics. Furthermore, since solving the quadratic equations by factorization requires students to find factors rapidly, factoring simple quadratics becomes quite a challenge, while non-simple ones (i.e., ax2 + bx + c where a ^ 1) become harder still. Factoring quadratics can be considerably complicated when the leading coefficient or the constant term has many pairs of factors (Bossé & Nandakumar, 2005).

The research of Filloy & Rojano (1989) suggested that an equation such as with an expression on the left and a number on the right is much easier to solve symbolically than an equation such as. This is because the first can be 'undone' arithmetically by reversing the operation 'multiply by 3 and subtract 1 to get 5' by 'adding 1 to 5 to get and then dividing 6 by 3 to get the solution.

Meanwhile the equation cannot be solved by arithmetic undoing and requires algebraic operations to be performed to simplify the equation to give a solution. This phenomenon is called 'the didactic cut'. It relates to the observation that many students see the 'equals' sign as an operation, arising out of experience in arithmetic where an equation of the form is seen as a dynamic operation to perform the calculation, 'three plus four makes 7', so that an equation such as is seen as an operation which may possibly be solved by arithmetic 'undoing' rather than requiring algebraic manipulation (Kieran, 1981).

Lima & Healy (2010) classified an equation of the form 'expression = number' as an evaluation equation, because it involved the numerical evaluation of an algebraic expression where the input value of x could be found by numerical 'undoing', and more general linear equations as manipulation equations, because they required algebraic manipulation for their solution.

The data of Lima & Tall (2008) presented an analysis of Brazilian students' work with linear equations that did not fit either the didactic cut or the balance model. Their teachers had used an 'expert-novice' view of teaching and had introduced the students to the methodology that they, as experts, found appropriate for solving equations, using the general principle of 'doing the same thing to both sides' to simplify the equation and move towards a solution. However, when interviewed after the course, students rarely used the general principle. They did not treat the equation as a balance to 'do the same thing to both sides', nor did they show any evidence of the didactic cut.

According to Matz, (1980) and Payne & Squibb (1990), Our purpose is not simply to find and catalogue errors. Instead we seek to evolve a single theoretical framework that covers all three aspects: the didactic cut, the balance model and the problem with 'doing the same thing to both sides'. Such a theoretical framework should relate to both cognitive development and the emotional effects of the learning experience. To integrate these different aspects into a single framework, we begin with a theoretical construct that relates current learning to previous experience.

This offers a refined formulation of the original research into the didactic cut by Filloy & Rojano (1989), where many of the students were able to solve simple evaluation equations before being taught to solve equations using algebraic manipulation. The notion of an equation as a process of evaluation is supportive for solving evaluation equations but problematic for manipulation equations. Another observation made at the time is that the introduction of the algebraic technique in solving linear equations caused a loss in ability for some students to solve simple equations using arithmetic undoing. This loss in facility when faced with a new technique is common in mathematics learning.

For instance, Gray (1991) noted that some children introduced to column subtraction may make errors that did not occur when they performed the same operation using simple mental arithmetic. This is consistent with the absence of the didactic cut in the data.

According to Lima & Tall (2008). The students had been presented with a new formal principle for solving equations by 'doing the same thing to both sides'. This new principle was not generally implemented as intended, instead the students focused on shifting symbols with additional rules as procedural embodiments that treated both evaluation and manipulation equations in the same way. Thus the students performed the same type of operation in both cases and made the same sort of error.

Tall (2011) formulated a working definition of a crystalline concept as 'a concept that has a structure of relationships that are a necessary consequence of its context'. Such a concept has strong internal bonds that hold it together so that it can be considered as a single entity.

Just as Sfard (1991) spoke of 'condensing' a process from a sequence of distinct steps which we may interpret as a metaphor for transforming a gas that is diffuse to a liquid that can be poured in a single flow, we can think of 'crystallizing' as the transition that turns the flowing liquid into a solid object that can be manipulated in the hand, or, in mathematics, manipulated in the mind as an entity. This metaphor does not mean that a crystalline concept has uniform faces like a chemical crystal, but that it has strong internal bonds that cause it to have a predictable structure.

Van Merrienboer and Jeroen (2013) investigated the perspectives on problem solving and instruction. It was found that problem solving should not be limited to well structured problem solving but be extended to real life problem solving.

Tsai et al. (2012) analyzed visual attention for solving multiple choice science problems. Studies showed that successful problem solvers focused more on relevant factors while unsuccessful problem solvers experienced difficulties in decoding the problem, in recognizing the relevant factors and in self regulating concentration. Kuo et al. (2012) experimented a hybrid approach to promoting students web based problem solving competence and learning attitude. Results show that middle and low achievement students in the experimental group gained significant.

Kaye (1915) argued that it was natural to seek for traces of Greek influence in later works of art and mathematics. There is evidence now to suggest Greek philosophy may be linked to or of a possible Indian origin. The difficulty in translating of scripts in Sanskrit was another reason in that Sanskrit scripts had to be translated before mathematicians could appreciate their actual mathematical value.

Joseph's (1991) study outlined the possibilities that may have led to the development of the vast knowledge of mathematics that we enjoy today this includes the mathematical achievement of all major civilizations such as Mayan, Babylonian, and Chinese among others. In this study, the author explores Indian mathematics by briefly investigating early India and the Vedic period-during which much of the foundations were laid for religious, philosophical and mathematical views of the world. The Vedic achievements are analysed to highlight the abstract and symbolic nature of their mathematics that ultimately led to the development of symbolic algebra (AD 500).

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When introducing algebra the use of letters should be withheld until it is evident that learners are ready for their use, and teaching should recognise and prepare learners for the various uses of letters in algebra as the need arises (Harper, 1987; Stols, 1996).

Pyke (2003) has shown that the learners' use of "... symbols, words, and diagrams to communicate about their ideas each contribute in different ways to solving tasks". The structurality of geometry and the visual overview that it provides facilitate thinking and effective investigation (Sfard, 1995). For example, the formulae for determining the areas of squares and rectangles can be used to introduce algebraic expressions. Such an approach could help learners to make links between arithmetic and algebra. A teaching sequence which allowed students to develop a procedural (operational) meaning for algebraic expressions such as $4x + 4y$ was designed by Chahouh and Herscovics (Kieran, 1992).

Math through the Ages (Berlinghoff & Gouvea, 2004) is an excellent book from which to learn the history of some key mathematical ideas. The text focuses on a few main ideas, and expands upon them. Specifically, it provides interesting stories and histories on people. However, it does not show most of the actual work that was needed to derive the formulae and ideas presented.

On the other hand, Journey through Genius (Dunham, 1990) provides many of the proofs and derivations of formulae in addition to interesting background information. However in this book, the focus of each chapter is a specific theorem, rather than the evolution of a mathematical idea.

Hattie (2009) noted that fluency with prerequisite knowledge, even at a very early stage, was highly predictive of latter success. The key prerequisite concepts and processes necessary to engage meaningfully with quadratics include basic whole number fluency, fraction computation, linear algebraic procedures, and coordinate geometry. A key process in working with quadratics is solving or finding the x intercepts, should there be any.

In most curricula this has involved factorisation, the square root method, completing the square, and the use of the quadratic formula. Each of these techniques has its own advantages and disadvantages when it comes to teaching, learning, and applying. Research has shown that students and teachers shy away from some techniques and favour factorisation, generally using coefficients that are easy to factorise since students' ability to perform fractional and radical arithmetic has been reported as low (Bosse & Nandakumar, 2005).

The study adds to the literature by supporting the findings of previous researchers who have documented that student understanding of quadratics equations is a problem area (Bosse & Nandakumar, 2005; Vaiyavutjamai & Clements, 2006; Vaiyavutjamai et al., 2005; Zakaria et al., 2010).

According to Kirschner et al. (2006) it adds to the literature by helping to explain why this was the case, at least in one school. Students who struggled did so due to a combination of factors that became critical. Some of these included a lack of prerequisite concepts and processes associated with fractions and algebra conventions (index law conventions and understanding the meaning of solve). Cognitive load theorists (provide an explanation as to why these deficits become a critical hindrance to engagement with quadratics.

Many students in this study did not have the tools to factorise, and many of those who could did not understand the implications of the factorised form for finding the roots. Conceptual errors were invariably proceeded by procedural errors. Sometimes the two were intertwined (DETE, 2013).

There was evidence that teacher interpretation of curriculum guidelines resulted in overemphasis of symbolic factorisation processes. The results suggest a lack of alternative pedagogies such as links to geometric models recommended by some authors and the integrated use of graphs in contexual settings (Barnes, 1991; Howden, 2001; Norton, 2015).

This deficit was most obvious in lack of understanding of null factor law and various forms of quadratics. These results add to the findings of previous authors regarding too narrow a focus on factorisation (Bosse & Nandakumar, 2005).

The sample school in this research project was a coeducational high school in south-east Queensland in a community of mixed socioeconomic index. The school is typical of outersuburban schools according to MySchool data from the Australian Curriculum, Assessment and Reporting Authority (ACARA) (ACARA, 2012a). The sample included a Year, 2011 Mathematics B class of 25 students. In Queensland, Mathematics B is a calculus-oriented, advanced senior mathematics class that qualifies the students to study science-oriented subjects at university. All the students had studied quadratics in Year 2010 as consistent with the state and national curriculum (ACARA, 2012b; Department of Education, Training and Employment [DETE], 2013; Queensland Studies Authority, 2004).

Overemphasis on relatively simple factorisation is concerning as many quadratic equations cannot be factorised. Further, other methods that are more efficient or that develop conceptualisation may be neglected in teaching (Bosse & Nandakumar, 2005).

For example, factorisation with algebra tiles links quadratics with basic multiplication and division concepts via the area model of rectangles and squares (Howden, 2001).

Geometric models are useful in adding understanding in developing the quadratic formula via completing the square procedure (Norton, 2015).

Barnes (1991) suggested using graphing calculators to plot quadratics with no roots, one root, or two roots and linking this to the discriminate values.

Research suggests that teachers tend to avoid teaching alternative methods due to high instances of process skill errors with techniques such as the quadratic formula and completing the square (Zakaria et al., 2010). From this literature review, it is clear that there is a need for further research into the sources of students' difficulties with quadratic equations.

APOS theory will be used as theoretical framework to study the level of cognitive development of students who completed a precalculus course using a traditional lecture/recitation model, as discussed by Arnon et al. (2013).

APOS theory was chosen since it has been used to study student learning of a variety of different mathematical concepts and has proven to give important insights on students learning of mathematics has an annotated bibliography). Also, it has been tested in the classroom and has proven effective in promoting students' learning of different concepts and guiding the development of classroom activities by Arnon et al. (2013).

In APOS theory (Arnon, et al, 2013) an Action is a transformation of a mathematical object performed by an individual that the individual perceives as external. It may be a transformation where the individual is limited to following an explicit algorithm step by step or is limited to the rigid application of a memorized fact. An individual who is limited to performing actions when dealing with a problem situation that involves a particular mathematical notion is said to have an action conception of the mathematical notion. So for example, a student who needs to be given the quadratic formula or who has memorized the quadratic formula and is only able to think of using it when a quadratic equation is given in standard form, or who is unable to anticipate or discuss the nature of its solutions without explicitly computing the solutions would show behaviour consistent with an action conception of the quadratic formula. If the individual repeats an action and reflects on it, the action may be interiorized into a Process. The process is now perceived as internal, under control of the individual. An individual with a process conception of a mathematical notion may reflect on it without having to explicitly carry out all the steps of the transformation. A process may be reversed and it may be coordinated with other processes. For example, a student who can anticipate being able to use the quadratic formula to find solutions of a quadratic equation regardless of the form in which the quadratic equation is given, or who without prompting can use the discriminant to discuss the nature of the solutions, or who can relate the nature of the solutions to the graphical representation of the corresponding quadratic function, would be showing behaviour consistent with at least a process conception of the quadratic formula.

Algebra is a core component of mathematics curriculum, algebra serves as a gatekeeper to higher mathematics and many prestigious occupations, and on the grounds of equity, all students should have access to it (Ahmad & Shahrill, 2014; Lim, 2000; National

Council of Teachers of Mathematics [NCTM], 2006; Pungut & Shahrill, 2014; Sarwadi & Shahrill, 2014; Shahrill, 2009). According to Moses (2000), and Strong and Cobb (2000).

This research seeks to understand why and what makes students choose a certain method or strategy in solving problems in algebra. In solving algebraic problems provides one of the ways to assess students' understanding of a concept. If the reasons can be identified, then it should be easier to improve the students' understanding to solve similar algebraic problems in the future. The thinking strategies of the students in solving problems on three sub-topics in algebra, namely, changing the subject of a given formula, factorisation of quadratic expressions and solving quadratic equations using quadratic formula. Thinking strategies had been defined as processes that involve thoughtful and effective use of cognitive skills and strategies for a particular context or type of thinking task where individuals engage in activating schemata and in integrating new subject matters into meaningful knowledge structures. In other words, thinking strategies refer to the processes by which individuals try to find solutions to problems through reflection (Davis, 1992).

Resnick (1982) stated "difficulties in learning are often a result of failure to understand the concepts on which procedures are based". Thus, it is important for teachers to develop insights into student thinking in order to identify students' difficulties and errors in understanding in algebra.

The framework of three worlds of mathematics is an overall theory of cognitive and affective growth in mathematics that has evolved to build from the early development of ideas in the child, through the years of schooling and on to the boundaries of research in formal mathematics (Tall, 2004, 2013). It is strongly related to a wide range of theoretical frameworks formulated by Piaget (1970), Dienes (1960), Bruner (1966), Van Hiele (1986), Skemp (1979), the SOLO taxonomy of Biggs & Collis (1982), the structural and operational mathematics of Sfard (1991), process-object theories (such as those of Sfard (1991), Dubinsky (Asiala et al., 1996), Gray & Tall (1994)), theories of advanced mathematical thinking (Tall (ed.), 1991), as well as theories from cognitive science such as the embodied theory of Lakoff and his colleagues (Lakoff & Núñez, 2000), the blending of cognitive structures formulated for example by Fauconnier and Turner (2002) and other aspects such as the role of various levels of consciousness (Donald, 2001).

Detailed discussion of all these aspects can be found. However, the main purpose of the theoretical framework is not to collate all these theories together with all their intricate details that differ in many ways, but to seek the fundamental essence of essential ideas that they have in common Tall (2013).

According to Lim (2000), students have a choice of either a rote-learned cross-multiplication method or a rotelearned grouping method when factorising a quadratic expression; however, neither was ever related to the distribution law. The selection of the method really depended on what their teachers preferred their students to use. Students remained unable to discover the factor of an algebraic expression, even at the post-teaching stage of factorising an algebraic expression.

Kotsopoulos (2007) stated that quadratic relations are one of the most conceptually challenging aspects of the high school curriculum. This is because many secondary students have difficulty with basic multiplication table fact retrieval. Since factorisation is a process of finding products within the multiplication table, this directly influences students' ability to engage effectively in factorisation of quadratics.

According to (Kotsopoulos (2007) most secondary school students and many university students were found to be confused about the concept of a variable and the meaning of a solution to a quadratic equation. For example, even if most students were able to obtain the correct solutions, $x = 3$ and $x = 5$, students thought that the two x's in the equation $(x - 3)(x - 5) = 0$ stood for different variables.

The students lack relational understanding and relied only on rote learning (Law & Shahrill, 2013; Pungut & Shahrill, 2014; Sarwadi & Shahrill, 2014; Vaiyavutjamai, 2004; Vaiyavutjamai, Ellerton & Clements, 2005; Vaiyavutjamai & Clements, 2006). In addition, when students were asked to solve $(x - a)(x - b) = 0$, they first expanded the linear expressions and then factorised before finally finding the solutions to that equation. This showed that the students lack understanding of the distributive law which, from a mathematical standpoint, is fundamental not only to the process of factorisation in algebra, but also to the reverse process of 'expanding brackets' (Lim, 2000).

In some cases, secondary students were expected to memorise the quadratic formula and to be able to apply it to solve quadratic equations despite not being taught how this formula could be derived (Lim, 2000). Thus students developed a perception that their main task was only to gain knowledge and to be able to solve quadratic equations using the quadratic formula; there was no real need to really understand why the method works. There are common reasons why students are unable to solve quadratic equations using the quadratic formula (Oliver, 1992). For example, he may not possess the required schema, or, his retrieval mechanism cannot locate his appropriate schema, or, the retrieved schema is flawed, incomplete or inappropriate (Abdullah, Shahrill & Chong, 2014; Chong & Shahrill, 2014; Shahrill & Abdullah, 2013).

As the solution of the problem is wholly determined by the combined information of the used cues and the content and structure of the retrieved schema, the solution will be wrong if the quadratic formula in the schema was flawed. In other words, the schema mediates the solution. On the other hand, changing the subject of a given formula plays an important role in mathematics. It is applied in various mathematical topics including function and its inverse and trigonometry. However, Lim (2000) found that students attempting to solve these equations still used descriptions of doubtful educational worth.

At the general intra- stage some operational actions are possible, but there is an absence of relationships between properties. At the inter- stage, the identification of relations between different processes and objects, and transformations are starting to form, but they remain isolated. The trans- stage is defined in terms of the construction of a synthesis between them to form a coherent structure (Cooley, Trigueros & Baker, 2007).

For example, in the genetic decomposition that we are about to describe, different processes and objects for solving quadratic equations using square roots, completion of square, quadratic formula, factoring, and graphical interpretation are given. The stage of development (intra-, inter-, trans-) of the schema of quadratic equations is a measure of the degree of interconnectedness of these ideas in the students' minds. The progression from action, to process, to object, and to having such constructions organized in schemas is a dialectical progression where there may be passages and returns from one type of construction to the other (Czarnocha, Dubinsky, Prabhu & Vidakovic, 1999).

What the theory states is that a student's tendency to deal with problem situations in diverse mathematical tasks involving a particular mathematical concept is different depending on whether the student understands the concept as an Action, a Process, or an Object or has constructed a coherent Schema. Hence an individual's mental construction of a particular mathematical concept may be classified (as action, process, object, intra-schema, inter-schema, or trans-schema) by inference made from observations of his/her overall behaviour when using or applying the mathematical concept in a diverse group of problem situations. In APOS theory, research starts with a conjecture of the mental constructions (in terms of the constructs of the theory) that students may do in order to understand a particular mathematical concept. The conjecture, called a genetic decomposition, is based on the mathematical concept itself, on the classroom experience of the researchers, and results from any available data. The conjecture is then tested by doing student interviews. What typically happens is that students will give evidence of doing some unexpected mental constructions and will also show difficulty on some of the conjectured constructions.

This leads to refining the genetic decomposition to better reflect the mental constructions that students actually do and it also leads to the design of student activities and more effective pedagogies to help them make particular constructions where they have shown difficulty. This marks the end of a research cycle and the beginning of the next one which would start with the class testing of the specially designed activities. Iterations of this cycle of research continue until stability is reached, that is, a genetic decomposition is obtained that serves to predict the mental constructions that students can actually do to understand the mathematical concept and also serves as a guide for the instruction of the particular mathematical concept. Our study is the first cycle of an APOS based research project dealing with student understanding of quadratic functions. The design of didactic material based on the refined genetic decomposition and its classroom implementation is not discussed.

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The understanding of quadratic equations with one unknown is fundamental for advanced studies in mathematics and other sciences. Nevertheless, it has been found in various investigations that many secondary school students and even undergraduate students do not truly understand these equations or the rules they use to solve them. For example, as Didis, Baş, and Erbaş (2011) concluded: ‘although students knew some rules related to solving quadratics, they applied these rules thinking about neither why they did so, nor whether what they were doing was mathematically correct. It was concluded that the students’ understanding in solving quadratic equations is instrumental (or procedural), rather than relational (or conceptual).’ Hence it is important to study how students learn to solve quadratic equations so that instruction in this topic may be improved. Different explanations for the scarce understanding of quadratic equations have been suggested.

Sönnerhead (2009), for example, noticed that mathematics textbooks in Sweden omit important concepts that would not be presented by many teachers, thus students will tend to develop a disconnected and incomplete set of ideas regarding quadratic equations.

Lima and Tall (2010) interviewed students who were taught to solve quadratic equations with a strong emphasis on using the quadratic formula as a general solution method for any type of equation. They conclude: ‘such a strategy enabled a small number of students to be able to solve specific quadratic equations, but it did not help in general to encourage students to construct flexible meanings in algebra’. A flexible understanding would allow adjusting the solution method to the type of quadratic equation and would require, as recommended by Kotsopoulos (2007), using different types of quadratic equations, not only on standard form 2 0= + +ax bx c, but also on factored form = − −1 2 a x r x r ( ) ( ) 0, and vertex form, 2 = + −a x h k ( ) 0.

Moreover, Olteanu and Holmqvist (2012), when comparing differences in student learning of the quadratic formula as a result of differences in their teaching, observed that a more successful teacher gave students different opportunities to experience variations in the form of the quadratic equation, and to discern the way in which the parts of equations are related to each other. Further support for this idea is provided by Eraslan (2005) when in his discussion of quadratic functions he argued that students have difficulty relating the vertex form of a quadratic equation to its standard form, preferring the latter form.

On the other hand, Bosse and Nandakumar (2005) observed that the probability that a given quadratic with integer coefficients in the interval]10,10−[has rational roots is only 15% and that this gets smaller as the interval of possible coefficients increases. Hence, it is unlikely that a quadratic equation resulting from an application of science or used to solve a real world problem can be solved using factoring techniques. They argue that this is reason enough to emphasize the other techniques of square roots, completing the square, and the quadratic formula.

Furthermore, Gray and Thomas (2001), reported on an experiment where students, who had received lessons using the graphing calculator, showed difficulty relating processes for the graphical and symbolic solution of quadratic equations. Some investigations on student understanding of quadratic equations refer to specific misconceptions. Vaiyavutjamai, Ellerton, and Clements (2005), Bosse and Nandakumar (2005), and Ochoviet and Oktaç (2009, 2011) observed, for example, that some students believe that the variable in an equation of the form = + +( ) ( ) x a x b c may have different values at once, which means that many students are not aware of the relation of the solution with the original quadratic equation. Also, as cited in the Vaiyavutjamai et al (2005), many of the students did not realize that quadratic equations often had two solutions.

In addition, Vaiyavutjamai et al (2005) and also, Tall, Lima, and Healy (2014) found that most of the students in their study could not find the correct solutions. 9=x by using correct procedures or correct explanations. Most of the students either they only found one solution.

Tall et al (2014) argued that students were for the most part shifting symbols around in a procedural embodied sense rather than using the more general reasoning of ‘doing the same on both sides’. Also, Didis et al (2011) observed that students were not aware of the missing root 0 when cancelling an in the equation 2 =2 3 x x and generally did not show a good understanding of the zero product theorem, a fact also observed by Ochoviet and Oktak (2009, 2011), and Bosse and Nandakumar (2005).

All of these investigations have their own context, with many of them involving observations made with pre-university students. In Puerto Rico, an unincorporated territory of the United States, the Department of Education establishes that the 8th grade students will learn how to solve simple quadratic equations by factorization and the zero product property, but it is not until 10th grade that students learn how to solve quadratic equations, not only simple quadratic equation, by the following techniques: factoring, the square root method, completing the square, the quadratic formula, and using technology. Hence, Puerto Rican students are expected to have seen quadratic equations in two different stages in their respective schools before beginning university studies. Given the context of the Puerto Rican beginning university student, this study proposes to investigate student understanding of quadratic equations by:

(1) Establishing a conjecture of the mental constructions (stated in terms of the constructs of APOS theory, as will be discussed further along) that beginning university students may do in order to understand how to solve quadratic equations.

(2) Using semi-structured interviews in order to investigate which of the conjectured mental constructions students can do and which they have difficulty doing.

(3) Using written work from more advanced undergraduate students to investigate their use and understanding of two specific mental constructions conjectured in the genetic decomposition.

The study of quadratic equations acts as a gateway to more advanced study of algebra and is a topic area that challenges many students (Bosse & Nandakumar, 2005; Vaiyavutjamai & Clements, 2006; Vaiyavutjamai, Ellerton, & Clements, 2005; Zakaria, Ibrahim, & Maat, 2010). Failure at working with quadratic equations virtually precludes students from accessing the powerful mathematics that is necessary to enrol in courses involving the study of sciences at tertiary levels (Watt, 2005). Despite the importance of this topic area there has been little research to inform the reform of pedagogy associated with quadratics.

The resounding theme in mathematics education research is that students' performance in the domain of quadratic equations is exceptionally poor and does not significantly increase even after instruction (Chaysuwan, 1996; Vaiyavutjamai et al., 2005). Students have been found to struggle particularly solving for x in the form $x\,2 = k$ (k>0) and $(x - r)\,(x - s) = 0$ where r and s are any real numbers (Vaiyavutjamai et al., 2005). The most concerning of all the data was that, out of a subsample of 29 second-year university students in the United States who were preserves middle-school mathematics specialist teachers, only 37% and 78% respectively could answer the two questions correctly (Vaiyavutjamai et al., 2005). Other than studies by the researchers noted above, there is a deficit in research and empirical evidence regarding students' performance with respect to solving quadratic equations. It is also important to consider the impact and current evidence relating to teaching methods and the learning of quadratic equations. Kotsopoulos (2007) reported that students need to develop procedural and conceptual knowledge through various learning experiences in an integrated manner. The Australian Academy of Science (AAS) also recognizes the intertwined relationship between conceptual understanding, procedural fluency, and problem solving and reasoning due to the hierarchical nature of mathematics (AAS, 2015).

The cognitive load work by Kirschner, Sweller, and Clark (2006) gives an explanation for the necessity of fluency with prerequisite knowledge. Without prerequisite fluency, short-term memory becomes overloaded and unable to effectively process the new concepts being learned. Hattie (2009) noted that fluency with prerequisite knowledge, even at a very early stage, was highly predictive of latter success. The key prerequisite concepts and processes necessary to engage meaningfully with quadratics include basic whole number fluency, fraction computation, linear algebraic procedures, and coordinate geometry. A key process in working with quadratics is solving or finding the x intercepts, should there be any. In most curricula this has involved factorisation, the square root method, completing the square, and the use of the quadratic formula.

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