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Report and Opinion

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On the Fundamental Theorem in Arithmetic Progression of Primes

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Abstract: Using Jiang function we prove the fundamental theorem in arithmetic progression of primes [1-3]. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetic progressions. Szemeredi theorem and Green-Tao theorem are absolutely false. They do not understand the arithmetic progression of primes [4-15] which is the greatest mathematical scandals of the world.

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Theorem. The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots,k-1, \qquad (1)$$

where $\omega_g = \prod_{2\le P\le P_g} P$ is called a common difference, P_g is called g -th prime.

We have Jiang function [1-3]

$$J_2(\omega) = \prod_{3\le P}(P-1-X(P)), \qquad (2)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{i=1}^{k-1}(q+\omega_g i) \equiv 0(\bmod P), \qquad (3)$$

where $q = 1,2,\cdots,P-1$.

If $P|\omega_g$, then $X(P)=0$; $X(P)=k-1$ otherwise. From (3) we have

$$J_2(\omega) = \prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-k). \qquad (4)$$

If $k = P_{g+1}$ then $J_2(P_{g+1}) = 0$, $J_2(\omega) = 0$, there exist finite primes P_1 such that $P_2,\cdots,P_k$ are primes. If $k < P_{g+1}$ then $J_2(\omega) \ne 0$, there exist infinitely many primes P_1 such that $P_2,\cdots,P_k$ are primes. The primes contain only $k < P_{g+1}$ long arithmetic progressions, but the primes have no $k > P_{g+1}$ long arithmetic progressions. We have the best asymptotic formula [1-3]

$$\pi_k(N,2) = \left|\{P_1 + \omega_g i = \text{prime}, 0 \le i \le k-1, P_1 \le N\}\right|$$

$$= \frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}(1+o(1)), \qquad (5)$$

where $\omega = \prod_{2\le P} P, \phi(\omega) = \prod_{2\le P}(P-1)$, ω is called primorial, $\phi(\omega)$ Euler function.

Suppose $k = P_{g+1} - 1$. From (1) we have

$$P_{i+1} = P_1 + \omega_g i, i = 0,1,2,\cdots,P_{g+1} - 2. \quad (6)$$

From (4) we have [1-2]

$$J_2(\omega) = \prod_{3 \le P \le P_g} (P-1) \prod_{P_{g+1} \le P} (P - P_{g+1} + 1) \to \infty \text{ as } \omega \to \infty \quad (7)$$

We prove that there exist infinitely many primes P_1 such that $P_2, \cdots, P_{P_{g+1}-1}$ are primes for all P_{g+1}.
From (5) we have

$$\pi_{P_{g+1}-1}(N,2) = \prod_{2 \le P \le P_g} \left(\frac{P}{P-1}\right)^{P_{g+1}-2} \prod_{P_{g+1} \le P} = \frac{P^{P_{g+1}-2}(P - P_{g+1} + 1)}{(P-1)^{P_{g+1}-1}} \frac{N}{(\log N)^{P_{g+1}-1}}(1+o(1)). \quad (8)$$

From (8) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N,2) > 1$ for large P_{g+1}.
Theorem is foundation for arithmetic progression of primes

Example 1. Suppose $P_1 = 2, \omega_1 = 2, P_2 = 3$. From (6) we have the twin primes theorem

$$P_2 = P_1 + 2. \quad (9)$$

From (7) we have

$$J_2(\omega) = \prod_{3 \le P} (P-2) \to \infty \text{ as } \omega \to \infty, \quad (10)$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From (8) we have the best asymptotic formula

$$\pi_2(N,2) = 2 \prod_{3 \le P} \left(1 - \frac{1}{(P-1)^2}\right) \frac{N}{\log^2 N}(1+o(1)). \quad (11)$$

Twin prime theorem is the first theorem in arithmetic progression of primes. Szemeredi and other mathematicians do not prove the twin prime theorem. Therefore Szemeredi theorem and Green-Tao are absolutely false [4-15]. The prime distribution is order rather than randomness. The arithmetic progressions of primes are not directly related to ergodic theory, harmonic analysis, discrete geometry and additive combinatorics. Conjectures and theorems on arithmetic progressions of primes are absolutely false [4-15], because they do not understand the arithmetic progressions of primes.

Example 2. Suppose $P_2 = 3, \omega_2 = 6, P_3 = 5$. From (6) we have

$$P_{i+1} = P_1 + 6i, i = 0,1,2,3. \quad (12)$$

From (7) we have

$$J_2(\omega) = 2 \prod_{5 \le P} (P-4) \to \infty \text{ as } \omega \to \infty, \quad (13)$$

We prove that there exist infinitely many primes P_1 such that P_2, P_3 and P_4 are primes. From (8) we have the best asymptotic formula

$$\pi_4(N,2) = 27 \prod_{5 \le P} \frac{P^3(P-4)}{(P-1)^4} \frac{N}{\log^4 N}(1+o(1)). \quad (14)$$

Example 3. Suppose $P_9 = 23, \omega_9 = 223092870, P_{10} = 29$. From (6) we have

$$P_{i+1} = P_1 + 223092870i, i = 0,1,2,\cdots,27. \quad (15)$$

From (7) we have

$$J_2(\omega)=36495360\prod_{29\le P}(P-28)\to\infty \text{ as } \omega\to\infty, \quad (16)$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{28}$ are primes. From (8) we have the best asymptotic formula

$$\pi_{28}(N,2)=\prod_{2\le P\le 23}\left(\frac{P}{P-1}\right)^{27}\prod_{29\le P}\frac{P^{27}(P-28)}{(P-1)^{28}}\frac{N}{\log^{28}N}(1+o(1)). \quad (17)$$

From (17) we are able to find the smallest solutions $\pi_{28}(N_0,2)>1$.

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes:

$6171054912832631+366384\times\omega_{23}\times n$, for $n=0$ to 24.

Theorem can help in finding for 26, 27, 28,…, primes in arithmetic progressions of primes.

Corollary 1. Arithmetic progression with two prime variables

Suppose $\omega_g=d$. From (1) we have

$$P_1, P_2=P_1+d,\ P_3=P_1+2d,\cdots,P_k=P_1+(k-1)d,\ (P_1,d)=1. \quad (18)$$

From (18) we obtain the arithmetic progression with two prime variables: P_1 and P_2,

$$P_3=2P_2-P_1,\quad P_j=(j-1)P_2-(j-2)P_1,\quad 3\le j\le k<P_{g+1}. \quad (19)$$

We have Jiang function [3]

$$J_3(\omega)=\prod_{3\le P}[(P-1)^2-X(P)], \quad (20)$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=3}^{k}[(j-1)q_2-(j-2)q_1]\equiv 0(\text{mod } P), \quad (21)$$

where $q_1=1,2,\cdots,P-1;\ q_2=1,2,\cdots,P-1.$

From (21) we have

$$J_3(\omega)=\prod_{3\le P\le k}(P-1)\prod_{k<P}(P-1)(P-k+1)\to\infty \text{ as } \omega\to\infty. \quad (22)$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3,\cdots,P_k$ are primes for $3\le k<P_{g+1}$

we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\left|\{(j-1)P_2-(j-2)P_1=\text{prime},3\le j\le k,P_1,P_2\le N\}\right|$$
$$=\frac{J_3(\omega)\omega^{k-2}}{\phi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1)), \quad (23)$$

From (23) we have the best asymptotic formula

$$\pi_{k-1}(N,3)=\prod_{2\le P\le k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k<P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad (24)$$

From (24) we are able to find the smallest solution $\pi_{k-1}(N_0,3)>1$ for large $k<P_{g+1}$.

Example 4. Suppose $k=3$ and $P_{g+1}>3$. From (19) we have

$$P_3=2P_2-P_1. \quad (25)$$

From (22) we have

$$J_3(\omega)=\prod_{3\le P}(P-1)(P-2)\to\infty \quad \text{as} \quad \omega\to\infty, \tag{26}$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From (24) we have the best asymptotic formula

$$\pi_2(N,3)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1))=1.32032\frac{N^2}{\log^3 N}(1+o(1)). \tag{27}$$

Example 5. Suppose $k=4$ and $P_{g+1}>4$. From (19) we have

$$P_3=2P_2-P_1,\ P_4=3P_2-2P_1. \tag{28}$$

From (22) we have

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-3)\to\infty \quad \text{as} \quad \omega\to\infty, \tag{29}$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are primes. From (24) we have the best asymptotic formula

$$\pi_3(N,3)=\frac{9}{2}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \tag{30}$$

Example 6. Suppose $k=5$ and $P_{g+1}>5$. From (19) we have

$$P_3=2P_2-P_1,\ P_4=3P_2-2P_1,\ P_5=4P_2-3P_1. \tag{31}$$

From (22) we have

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-4)\to\infty \quad \text{as} \quad \omega\to\infty, \tag{32}$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are primes. From (24) we have the best asymptotic formula

$$\pi_4(N,3)=\frac{27}{2}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \tag{33}$$

Green and Tao study only **corollary 1**, which is not the theorem [4-9].

Corollary 2. Arithmetic progression with three prime variables

From (18) we obtain the arithmetic progression with three prime variables: P_1, P_2 and P_3

$$P_4=P_3+P_2-P_1,\ P_j=P_3+(j-3)P_2-(j-3)P_1,\ 4\le j\le k<P_{g+1} \tag{34}$$

We have Jiang function

$$J_4(\omega)=\prod_{3\le P}((P-1)^3-X(P)), \tag{35}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=4}^{k}(q_3+(j-3)q_2-(j-3)q_1)\equiv 0(\text{mod } P), \tag{36}$$

where $q_i=1,2,\cdots,P-1,\ i=1,2,3$.

Example 7. Suppose $k=4$ and $P_{g+1}>4$. From (34) we have

$$P_4=P_3+P_2-P_1. \tag{37}$$

From (35) and (36) we have

$$J_4(\omega)=\prod_{3\le P}(P-1)(P^2-3P+3)\to\infty \quad \text{as} \quad \omega\to\infty, \tag{38}$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that P_4 are primes. we have the best asymptotic formula

$$\pi_2(N,4)=2\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)). \tag{39}$$

For $k\ge 5$ from (35) and (36) We have Jiang function

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2$$
$$\times\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\to\infty$$

as $\omega\to\infty$. (40)

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that $P_4,\cdots,P_k$ are primes for $5\le k<P_{g+1}$.

we have the best asymptotic formula

$$\pi_{k-2}(N,4)=\left|\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},4\le j\le k,P_1,P_2,P_3\le N\}\right|$$
$$=\frac{J_4(\omega)\omega^{k-3}}{\phi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)). \tag{41}$$

From (41) we have

$$\pi_{k-2}(N,4)$$
$$=\prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)). \tag{42}$$

From (42) we are able to find the smallest solution $\pi_{k-2}(N_0,4)>1$ for large $k<P_{g+1}$.

Corollary 3. Arithmetic progression with four prime variables

From (18) we obtain the arithmetic progression with four prime variables: P_1,P_2,P_3 and P_4

$$P_5=P_4+2P_3-3P_2+P_1, \quad P_j=P_4+(j-3)P_3-(j-2)P_2+P_1,$$
$$5\le j\le k<P_{g+1} \tag{43}$$

We have Jiang function

$$J_5(\omega)=\prod_{3\le P}\left[(P-1)^4-X(P)\right], \tag{44}$$

$X(P)$ denotes the number of solutions for the following congruence

$$\prod_{j=5}^{k}[q_4+(j-3)q_3-(j-2)q_2+q_1]\equiv 0 \pmod P, \tag{45}$$

where

$$q_i=1,\cdots,P-1, i=1,2,3,4$$

Example 8. Suppose $k=5$ and $P_{g+1}>5$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1. \tag{46}$$

From (44) and (45) we have

$$J_5(\omega)=12\prod_{5\le P}(P-1)(P^3-4P^2+6P-4)\to\infty \text{ as } \omega\to\infty. \quad (47)$$

We prove there exist infinitely many primes P_1,P_2,P_3 and P_4 such that P_5 are primes.
We have the best asymptotic formula

$$\pi_2(N,5)=\frac{J_5(\omega)\omega}{\phi^5(\omega)}\frac{N^4}{\log^5 N}(1+o(1)). \quad (48)$$

Example 9. Suppose $k=6$ and $P_{g+1}>6$. From (43) we have

$$P_5=P_4+2P_3-3P_2+P_1,\quad P_6=P_4+3P_3-4P_2+P_1. \quad (49)$$

From (44) and (45) we have

$$J_5(\omega)=10\prod_{5\le P}(P-1)(P^3-5P^2+10P-9)\to\infty \text{ as } \omega\to\infty. \quad (50)$$

We prove there exist infinitely many primes P_1,P_2,P_3 and P_4 such that P_5 and P_6 are primes.
We have the best asymptotic formula

$$\pi_3(N,5)=\frac{J_5(\omega)\omega^2}{\phi^6(\omega)}\frac{N^4}{\log^6 N}(1+o(1)). \quad (50)$$

For $k\ge 7$ from (44) and (45) we have Jiang function

$$J_5(\omega)=6\prod_{5\le P\le(k-4)}(P-1)(P^2-3P+3)$$
$$\times\prod_{(k-4)<P}\left\{(P-1)^4-(P-1)^2\left[(P-3)(k-4)+1\right]-(P-1)(2k-9)\right\}\to\infty$$

as $\omega\to\infty$ (51)

We prove there exist infinitely many primes P_1,P_2,P_3 and P_4 such that $P_5,\cdots,P_k$ are primes.
We have best asymptotic formula

$$\pi_{k-3}(N,5)=\left|\left\{P_4+(j-3)P_3-(j-2)P_2+P_1=\text{prime},5\le j\le k,P_1,\cdots,P_4\le N\right\}\right|$$
$$=\frac{J_5(\omega)\omega^{h-4}}{\phi^k(\omega)}\frac{N^4}{\log^k N}(1+o(1)). \quad (52)$$

The greatest mathematical scandals of the world

Szemerédi theorem (1975): Any subset of the integers of positive density contains arbitrarily long arithmetic progressions [10] which is not directly related to the arithmetic progressions and is scandal. (1) mathematical scandal (Notices of the AMS, 55, 2008, 1284). In 2008 Endre Szemerédi has been awarded the Rolf Schock prize in mathematics by the Royal Swedish Academy of Science. Szemerédi was honored "for his deep and pioneering work from 1975 on arithmetic progressions in subsets of the integers, which has led to great progress and discoveries in several branches of mathematics." Szemerédi awarded Schock prize which is scandal. (2) Mathematical scandal (Notices of the AMS, 55, 2008, 486-487). The Steele prize in 2008 for a seminal contribution to mathematical research is awarded to Endre Szemerédi for the paper "on sets of integers containing no k elements in arithmetic progression", Acta Arithmetica XXVII (1975), 199-245. Szemerédi receive the Steele prize which is scandal. (3) Mathematical Scandal (Notices of the AMS, 54, 2007, 631-632). In 1977 using ergodic theory Harry Furstenberg proved Szemerédi theorem. He receive 2006-2007 Wolf prize which is scandal. (4) Mathematical scandal (Notices of the AMS, 55,2008,58). In 2004 Ben Green proved Szemerédi theorem. He receive 2007 Sastra Ramanujan prize which is scandal. (5) Mathematical scandal (Notices of AMS,53,2006,1041-1042). In 2004 Terence Tao proved Szemerédi theorem. He receive 2006 Fields medal which is scandal. (6) Mathematical scandal (Notices of the AMS,54,2007,48-49). In 2004 Terence Tao proved Szemerédi theorem. He receive 2006 Sastra

Ramanujan prize which is scandal. (7) Mathematical scandal. Annals of Math., GAFA., Adv. Math., Duke math. J., and other mathematical journals publish the false papers on Szemerédi theorem. (8) Mathematical scandal. Institute for advanced study-school of mathematics, Claymath institute and Max Planck institute for mathematics support the Szemerédi theorem and Green-Tao theorem which are scandal. (9) Mathematical scandal. They falsely understand the prime number theorem $N/\log N$, The density of primes is 1/log*N*. The prime distribution is the random. They do not recognize that the prime distribution is Jiang function. (10) Mathematical scandal. The New York Times (13 March 2007) described it this way: "In 2004, Dr. Tao, along with Ben Green, a mathematician now at the University of Cambridge in England, solved a problem related to the Twin Prime Conjecture by looking at prime number progressions-series of numbers equally spaced. (For example, 3, 7 and 11 constitute a progression of prime numbers with a spacing of 4; the next number in the sequence, 15, is not prime.) Dr. Tao and Dr. Green proved that it is always possible to find, somewhere in the infinity of integers, a progressions of prime numbers of equal spacing and any length."

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