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Conditioning And Preconditioning Of The Model Error Formulation: $\mathcal{J}$ (P)

*Dr. Rajeev Kumar and **Gurpreet Kaur

*Assistant Professor, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
**Research Scholar, Department of Mathematics, OPJS University, Churu, Rajasthan (India)
Email: shargun500@gmail.com; Contact No. +91-9996347359

Abstract: The theorems in this section extend the work of Haben et al. [41] on the condition number of the standard 3DVAR and 4DVAR systems. We briefly discussed and compared J **(p)** to the conventional sc4DVAR approach (2.8) in Section 5.1.2 using the lower and upper bounds derived in [41] and the bounds we have derived in Theorem 5.1.2. In sc4DVAR, ^{S} is the only error variance ratio, which means if the observations are accurate and/or the background error variance is large then the condition number of the of Hessian of the sc4DVAR problem would rise. The assumptions become more specific with each theorem. The first theorem assumes general correlation structures for the background, observation and model errors while assuming there are fewer observations than the dimension of state space. The second theorem derives bounds that are more specific to a particular class of covariance and model matrices, whereas the final theorem is specific to the advection equation. We then take the preconditioned Hessian of objective function J **(p)** and bound its condition number. We then show the improvement in overall conditioning and minimisation iteration rates of the preconditioned problem compared to the original problem.

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Introduction

Numerical weather prediction (NWP) centres produce forecasts of future weather states using a numerical model of the atmosphere to evolve an estimate of the initial state of the atmosphere forward in time. The accuracy of this estimate, called the analysis, is therefore a major factor in determining the accuracy of the resultant forecast. Variational data assimilation (Var) is one method popularly used in NWP centres for finding the analysis. In Var the analysis is the minimiser of a cost function. The cost function is essentially a weighted measure of the distance between the forecast states and the available observations within a fixed time window, weighted using the background (or forecast) and observation error covariance matrices. The resulting solution is the maximum likelihood best estimate of the state of the atmosphere under certain assumptions.

In the previous chapter we examined the e_ect that various assimilation parameters had on the iterative solution process of the wc4DVAR problem when applied to the 1D advection equation.

The results in this chapter extend the results in [41], where the author bounded the condition number of the 3DVAR Hessian and then the Hessian of the strong-constraint 4DVAR objective function, denoted as S. We have derived a general result linking the condition numbers of the sc4DVAR Hessian S and the wc4DVAR Hessian $\mathbf{S_p}$ such that,

$$\kappa(S) < \kappa(S_p), \qquad (5.1)$$

with no assumptions. This result shows that the condition number of the Hessian of sc4DVAR can *never* exceed the condition number of the Hessian of the wc4DVAR J (**p)** formulation for identical assimilation problems.

/ **E** *(HiMi,o)*^{T}*R*$^{-1}$*H*$_i$*M*$_i$*,o* **E** *(HiMi,*$_0$*)*^{T}R^{-1}H;M$_M$ **E** *(H*$_i$*M*$_i$*,0)*^{T}*R*$^{-1}$*H*$_i$*M*$_i$*,2* ... (fl,,M,,)$_0$)^{T}R^{-1}H,, \
i=0**'** **' i**=1 ' ' *i=2* '
E (H$_;$M$_M$)^T Rr^{1}H$_i$M$_{i0}$ **E** (H$_;$M$_M$)^T Rr^{1}H$_i$M$_{i1}$ **E** *(H*$_i$*M*$_{i>1}$*)*^{T} Rr^{1}H$_i$M$_{i2}$ ... (fl,,M,,$_i$1)^T R^{-1}H,,
i = 1 **'** * **' i** = 1 **'** * **' i**=2 *
n n n-2 .
E (H$_;$M$_{i>2}$)^T R7^{1}H$_i$M$_{i>0}$ **E** (H$_;$M$_{i>2}$)^T *R7*^{1}*H*$_i$*M*$_{i1}$ **E** (H$_;$M$_{i>2}$)^T *R*$^{\wedge 1}$*H*$_i$*M*$_{it2}$ . .
. .. . (H,,M,,,,,_1)^T R^{-1}H,,

$\backslash H^TR^{-1}H,,M,,,0 \quad H^T R^{--1}H,,M,,_{>}1 \quad ... \quad H^TR^{-1}H,,M,,,,,\_1\ H\hat{}R^{-1}H_n\ )$ (5.2)

and its preconditioned counter-part

$\mathbf{S_p = I + D^{1/2}L^{-T}\ H^T\ R^{-1}HL^{-1}D^{12}.}$ (5.3)

The Eigen value spectrum of these matrices are not explicitly known and in practice they are too computationally expensive to calculate explicitly. So we take the route of estimating the condition number of the Hessian by bounding it in order to obtain information from the expressions yielded by the bounds. We utilize the bounds to gain insight into the Hessian condition number sensitivities of the objective function J **(p)** and its preconditioned counter-part.

We first derive bounds on the condition number of the Hessian $\mathbf{S_p}$ with some simple assumptions on the observations. The assumptions become more specific with each theorem. The first theorem assumes general correlation structures for the background, observation and model errors while assuming there are fewer observations than the dimension of state space. The second theorem derives bounds that are more specific to a particular class of covariance and model matrices, whereas the final theorem is specific to the advection equation. We then take the preconditioned Hessian of objective function J **(p)** and bound its condition number. We then show the improvement in overall conditioning and minimisation iteration rates of the preconditioned problem compared to the original problem.

The insight gained from the bounds are demonstrated through numerical experiments on the condition number. We also further demonstrate the condition number sensitivities obtained from the bounds by examining their effect on the convergence rate of the model error estimation an preconditioned model error estimation minimisation problems.

$\hat{}\ E\ \mathbf{(X«)\ (Xa)^j}\ (Xa)^j\ v^H\ H^T\ Hvk + \bullet\bullet\bullet + \bullet\bullet\bullet + \mathbf{(A«)^n v^H\ H^T\ Hvk + (X«)^n v^H\ H^T\ HVk}$

i=1 j=0 **(5.17)**

where the first term in the *geometric* series (5.17) comes from the main diagonal of (5.13). The second term of (5.17) is from the upper off-diagonal block entries of (5.13) and the third term is from the lower off-diagonal block entries. This pattern continues until the final term in the bottom right hand corner of (5.13), which coincides with the final term in (5.17).

We now compute each of the terms in the series above. We have

$v^H\ H^T\ Hvk = N$ (5.18)

since circulant matrices have orthogonal eigenvectors and H^TH is a square matrix with q unit entries on the main diagonal at positions of observation and 0 elsewhere. We also know the following to be true:

$(X«)\ (X«) = |X«|^q.$ (5.19)

Comparison to Strong-Constraint 4DVAR

The bounds in Theorem 5.1.2 bear some similarities to the bounds derived on the condition number of the Hessians of the sc4DVAR and 3DVAR problems as shown in [41] (Theorem 6.1.2 and Theorem 7.1.2). The influence of the condition number of B_0 on the condition number of the sc4DVAR Hessian is similar to the influence of the condition number of D on the condition number of S_P. The B_0 matrix was influenced only by the condition number of the background error covariance matrix C_B, whereas D is influenced by C_B, CQ and the ratio of o_b/o_q. We further illustrate this by taking a simplified scenario as an example.

Numerical Results

We now demonstrate the bounds through numerical experiments. We also highlight sensitivities of the condition number of **S p** with respect to assimilation parameters, which have been revealed by the theorems in Section 5.1.

We let **M** be the linear advection model as in (3.71), with a one-dimensional domain of size **N** = 500 grid points and spatial intervals of A**x** = 0.1. We use temporal intervals of A**t** = 0.1 and wave speed ***a*** = —0.3. We let **n** = 2, so we have a total of three model time levels including initial time, all of which are observed. We let **q** = 20 spatial observations at the grid points with equal spacing, so ***q (n*** + 1) = 60. The temporal observations are made every 3 model time steps, so at $t_0 = 0$, $t\backslash = 3At$ and $t_2 = 6At$. We assume no spatial correlations for the observation errors whereas the background and model errors are spatially correlated (as in Sections 3.3.4.1 and 3.3.4.2), $\mathbf{B_0} = ob C_{SOAR}$, $Q_i = \mathbf{Q} = \mathbf{oq^2} C_{LAP}$, $R = o^2{}_Q I_q$ where $o_b = \mathbf{o_q} = o_o$ =1 unless otherwise stated. We denote the correlation length-scale of a covariance matrix **C** as ***L (C).***

Conclusion:

We have obtained new general bounds on the condition number of the wc4DVAR J **(p)** formulation. We then developed the bounds by making simple assumptions about the observations, the nature of the model and the covariance matrices. This was then extended to the specific case where the model is a 1D advection equation, which is of relevance in NWP since advection is a physical process occurring in numerous models describing atmospheric systems.

The theorems in this section extend the work of Haben et al. [41] on the condition number of the standard 3DVAR and 4DVAR systems. We briefly discussed and compared J **(p)** to the conventional sc4DVAR approach (2.8) in Section 5.1.2 using the lower and upper bounds derived in [41] and the bounds we have derived in Theorem 5.1.2. In sc4DVAR, S is the only error variance ratio, which means if the observations are accurate and/or the background error variance is large then the condition number of the of Hessian of the sc4DVAR problem would rise. We showed that for wc4DVAR there is an intricate balance to be considered for the combination of the three ratios, —, — and —. We showed that the magnitude.

' **Sq** ' So So °

(whether small or large) of the difference between the error variances in wc4DVAR directly effects the condition number of $\mathbf{S_p}$.

The bounds in Theorem 5.1.2 also indicated the sensitivity of κ (S_P) to correlation length-scales of the background and model error covariance matrices since these have a direct influence on κ (D) and hence κ (S_P). We have also shown for the advection equation in Theorem 5.1.3, that the assimilation window length, n, influences the condition number of $\mathbf{S_p}$.

Corresponding author:
Mrs. Gurpreet Kaur
Research Scholar, Department of Mathematics,
OPJS University, Churu,
Rajasthan (India)
Contact No. +91-9996347359
Email- Email: shargun500@gmail.com

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