The k-out-of-n system model with degradation facility

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Abstract: In this paper, we study the reliability analysis of K-out-of-N system with degradation facility. Let failure rate, degradable rate and repair rate of components are assumed to be exponentially distributed. There are two types of repair. The first is due to failed state. The second is due to degraded state. The expressions of reliability and mean time to system failure are derived with repair and without repair. We used several cases to analyze graphically the effect of various system parameters on the reliability system.
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Keywords: Reliability, Mean time to system failure, Simplex system.

1. Introduction

The general structure of series and parallel systems: the so-called k-out-of n system. In this type of system, if any combinations of k units out of n independent units work, it guarantees the success of the system. For simplicity, assume that all units are identical. Furthermore, all the units in the system are active. The parallel and series systems are special cases of this system for k= 1 and k = n, respectively.

In this paper, we provide a detailed coverage on reliability evaluation of the k-out-of-n systems with degradation. We study the simple model "one unit with one degradable state (simplex)" then we investigate triple modular redundancy (TMR) without repair components. Finally we study the k-out-of-n in details with repair and without repair of components. In addition, we perform numerical results to analyze the effects of the various system parameters on the system reliability.

1.1 Notations

n: Number of components in the system.
k : Minimum number of components that must work for the k-out-of-n system to work
λ_1: The failure rate of the unit.
λ_2: The degradable rate of the unit.
μ_1: The repair rate of failed unit.
μ_2: The repair rate of degraded unit.
$P_{i,j}(t)$: Probability that there are i degradable units and j failed units in the system at time t where $i = 0,1,2,\dots,n$, $j = 0,1,2,\dots,n-k+1$.
s : Laplace transform variable.
$P^*_{i,j}(t)$: Laplace transform of $P_{i,j}(t)$.

$P_i(t)$: Probability for i = 0, 1, 2

0→operable (normal) state

1→ degradable state

2→ failed state

$P_i^*(s)$: Laplace transform of $P_i(t)$.

$R_{sim}(t)$: Reliability function of simplex system with degradation

$MTTF_{sim}$: Mean time to failure of simplex system with degradation

$R_{tmr}(t)$: Reliability function of TMR system with degradation

$MTTF_{tmr}$: Mean time to failure of TMR system with degradation

$R_N(K,N)$: Reliability of k-out-of-n system or probability that at least k out of the N components are working (nonrepairable system), where 0 ≤ k ≤ n and both k and n are integers.

$R_R(k,n)$: Reliability of k-out-of-n system or probability that at least k out of the N components are working (repairable system), where 0 ≤ k ≤ n and both k and n are integers.

$MTTF_R(k,n)$: Mean time to failure of a k-out-of-n system.

μ_1 : The repair rate of failed unit.

μ_2 : The repair rate of degraded unit.

$\mu_{1,n}$: Mean repair rate when there are n failed units in the system

$\mu_{2,n}$: Mean repair rate when there are n degraded units in the system.

1.2 Nonrepairable k-out-of-n System

1.2.1. Simplex system

We get Simplex system when $n = k = 1$

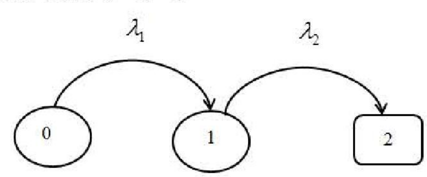

Figure1-1. Simplex system with degradation diagram.

Based on the state transition diagram in Figure1-1, we can derive the following differential equations:

$$\frac{dP_0(t)}{dt} = -\lambda_1 P_0(t) \quad (1)$$

$$\frac{dP_1(t)}{dt} = -\lambda_2 P_1(t) + \lambda_1 P_0(t) \quad (2)$$

$$\frac{dP_2(t)}{dt} = -P_2(t) + \lambda_2 P_1(t) \quad (3)$$

Initial conditions:

$$P_i(0) = \begin{cases} 1, & where \quad i = 0 \\ 0, & otherwise \end{cases}$$

Taking the Laplace transform of equations $(1-3)$ and applying initial conditions, we have

$$(s+\lambda_1)P_0^*(s)=1 \quad (4)$$

$$(s+\lambda_2)P_1^*(s)-\lambda_1 P_0^*(s)=0 \quad (5)$$

$$(s)P_2^*(s)-\lambda_2 P_1^*(s)=0 \quad (6)$$

Using the Laplace transform technique, the solutions of $P_i^*(s)$, i=0, 1, 2 are given

$$P_0^*(s)=\frac{1}{(s+\lambda_1)} \quad (7)$$

$$P_1^*(s)=\frac{\lambda_1}{s^2+s(\lambda_1+\lambda_2)+\lambda_1\lambda_2} \quad (8)$$

$$P_1^*(s)=\frac{\lambda_1\lambda_2}{s(s^2+s(\lambda_1+\lambda_2)+\lambda_1\lambda_2)} \quad (9)$$

Inverse Laplace transforms of these equations yield

$$P_0(t)=e^{-\lambda_1 t} \quad (10)$$

$$P_1(t)=\frac{\lambda_1(e^{-\lambda_2 t}-e^{-\lambda_1 t})}{\lambda_1-\lambda_2} \quad (11)$$

The reliability function of the system can be written as

$$R_{sim}(t)=P_0(t)+P_1(t)=\frac{\lambda_1 e^{-\lambda_2 t}-\lambda_2 e^{-\lambda_1 t}}{\lambda_1-\lambda_2} \quad (12)$$

The MTTF can be obtained from this equation

$$MTTF_{sim}=\frac{1}{\lambda_1}+\frac{1}{\lambda_2} \quad (13)$$

1.2.2. Triple Modular Redundancy: TMR

In this scheme, three identical redundant units or modules perform the same task simultaneously with degradable rate. The TMR system only experiences a failure when more than one component fails. In other words, this type of redundancy can tolerate failure of a single component. Figure 1-2 shows a diagram of the TMR scheme.

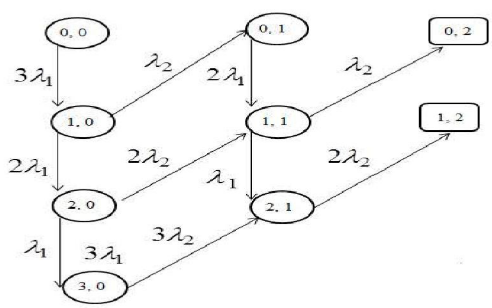

Figure1-2. TMR with degradation diagram.

From Figure1-2, taking LaPlace transforms of the state equations yields:

$$(s+3\lambda_1)P_{0,0}^*(s)=1 \quad (14)$$

$$(s+2\lambda_1)P_{0,1}^*(s)-\lambda_2 P_{1,0}^*(s)=0 \quad (15)$$

$$(s+2\lambda_1+\lambda_2)P_{1,0}^*(s)-3\lambda_1 P_{0,0}^*(s)=0 \quad (16)$$

$$(s+\lambda_1+\lambda_2)P_{1,1}^*(s)-2\lambda_1 P_{0,1}^*(s)-2\lambda_2 P_{2,0}^*(s)=0 \quad (17)$$

$$(s+\lambda_1+2\lambda_2)P_{2,0}^*(s)-2\lambda_1 P_{1,0}^*(s)=0 \quad (18)$$

$$(s+2\lambda_2)P_{2,1}^*(s)-\lambda_1 P_{1,1}^*(s)-3\lambda_2 P_{3,0}^*(s)=0 \quad (19)$$

$$(s)P_{0,2}^*(s)-\lambda_2 P_{1,1}^*(s)=0 \quad (20)$$

$$(s)P_{1,2}^*(s)-2\lambda_2 P_{2,1}^*(s)=0 \quad (21)$$

Solving equations (14-21) and taking inverse Laplace transforms of these equations, we get the reliability function of the system

$$R_{tmr}(t)=\sum_{i,j=0}^{i=3,j=1} P_{i,j}(t)$$

$$R_{tmr}(t)=\frac{3\left(\lambda_1^2 e^{-2\lambda_2 t}+\lambda_2^2 e^{-2\lambda_1 t}-2\lambda_1\lambda_2 e^{-(\lambda_1+\lambda_1)t}\right)}{\lambda_1-\lambda_2}$$

$$-\frac{2\left(\lambda_1^3 e^{-3\lambda_2 t}-\lambda_2^3 e^{-3\lambda_1 t}+3\lambda_1\lambda_2^2 e^{-(\lambda_1+2\lambda_1)t}-3\lambda_1^2\lambda_2 e^{-(\lambda_1+2\lambda_2)t}\right)}{\lambda_1-\lambda_2} \quad (22)$$

As we know the reliability of the triple modular redundancy of a component with one failure rate can be obtained from this equation

$$R_{TMR}(t)=3R^2-2R^3 \quad (23)$$

Where $R=e^{-\lambda t}$ reliability of a component with failure rate λ, from (22) we get

$$R_{tmr}(t)=3(R_{sim})^2-2(R_{sim})^3 \quad (24)$$

The $MTTF_{tmr}$ can be obtained from this equation

$$MTTF_{tmr}=\int_0^{\infty} R_{tmr}(t) \quad (25)$$

1.2.3. General system

We will examine a general model for analysis of such systems when they are nonrepairable. When a k-out-of-n system is put into operation, all n components are in good condition. The system is failed when the number of working components goes down below k or the number of failed components has reached n−k+1. We consider the components in a k-out-of-n system are i.i.d. Let the failure rate and degradable rate occur independently of the states of other units and follow exponential distributions with λ_1, λ_2, respectively.

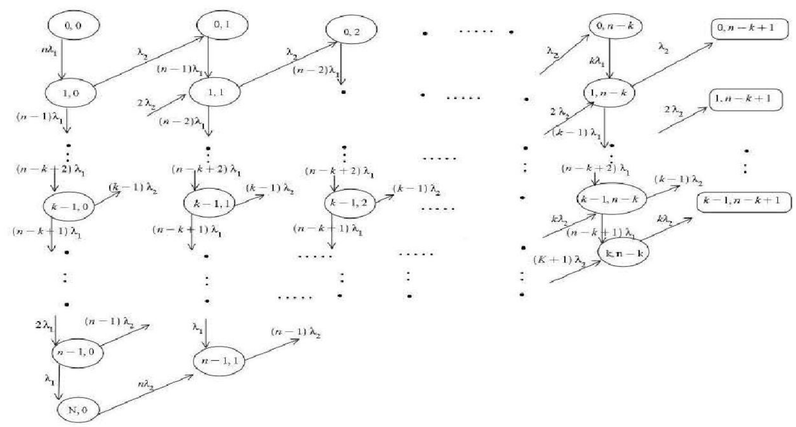

Figure1-3. State-transition-rate diagram of Nonrepairable System with two types of failure

At time $t=0$ the system starts operation with no failed units. The Laplace transforms of $P_{i,j}(t)$ are defined by:

$$P^*_{i,j}(s)=\int_0^\infty e^{-st}P_{i,j}(t)\,dt,\quad i=0,1,2,\ldots,n. and\ j=0,1,2,\ldots,n-k+1.$$

Based on the model descriptions, the system state transition diagram is given in Fig.1-3 and it leads to the following Laplace transform expressions for $P_{i,j}{}^*(s)$:

$$(s+n\lambda_1)P^*_{0,0}(s)=1 \tag{26}$$

$$(s+(n-i)\lambda_1)P^*_{0,j}(s)-\lambda_2P^*_{1,j-1}(s)=0,1\le i\le n-k \tag{27}$$

$$(s+(n-i)\lambda_1+i\lambda_2)P^*_{i,0}(s)-(n-i+1)\lambda_1P^*_{i-1,0}(s)=0,1\le i\le n-1 \tag{28}$$

$$(s+n\lambda_2)P^*_{n,0}(s)-\lambda_1P^*_{n-1,0}(s)=0 \tag{29}$$

$$(s+(n-i)\lambda_2)P^*_{n-i,i}(s)-(n-i+1)\lambda_2P^*_{n-i+1,i-1}(s)-\lambda_1P^*_{n-i-1,i}(s)=0,1\le i\le n-k \tag{30}$$

$$(s+(n-k-i)\lambda_2)P^*_{n-k-i,i}(s)-(n-k-i)\lambda_2P^*_{n-k-i+1,i-1}(s)$$
$$-(n-k-1)\lambda_1P^*_{n-k-i-1,i}(s)=0,1\le i\le n-k-1 \tag{31}$$

$$(s+(n-k-j+i)\lambda_2+(k-i)\lambda_1)P^*_{n-k-j+i,j}(s)-(k-i+1)\lambda_1P^*_{n-k-j+i-1,j}(s)$$
$$-(n-k-j+i+1)\lambda_2P^*_{n-k-j+i+1,j-1}(s)=0,1\le i\le k-1,1\le j\le n-k \tag{32}$$

$$(s+(i+1)\lambda_2+(n-j-i-1)\lambda_1)P^*_{i+1,j}(s)-(i+2)\lambda_2P^*_{i+2,j-1}(s)$$
$$-(n-j-i)\lambda_1P^*_{i,j}(s)=0,0\le i\le n-k-2,1\le j\le n-k-i-1 \tag{33}$$

$$(s)P^*_{i,n-k+1}(s)-(i+1)\lambda_2P^*_{i+1,n-k}(s)=0,0\le i\le k-1 \tag{34}$$

From solving equations $(26-34)$ and taking inverse Laplace, we obtain the reliability function as follows:

$$R_N(k,n)=L^{-1}\left(\sum_{i,j=0}^{\substack{i=n,\\ j=n-k+1}} P_{i,j}^*(s)\right)=\left(\sum_{i,j=0}^{\substack{i=n,\\ j=n-k+1}} P_{i,j}(t)\right) \tag{35}$$

$$R_N(k,n)=\sum_{i=K}^{n}\binom{n}{i}(R_{sim})(1-R_{sim})^{n-i} \tag{36}$$

The mean time to failure $MTTF_N(k,n)$ can be obtained from the following relation.

$$MTTF_N(k,n)=\lim_{s\to 0}R_r^*(s)=\lim_{s\to 0}\left\{\sum_{i,j=0}^{\substack{i=n,\\ j=n-k+1}} P_{i,j}^*(s)\right\} \tag{37}$$

When we perform a sensitivity analysis for changes in the $R_N(k,n)$ resulting from changes in system parameters λ_1 and λ_2 .By differentiating equation (36) with respect to λ_1 we obtain,

$$\frac{\partial R_N(k,n)}{\partial \lambda_1}=\frac{\partial}{\partial \lambda_1}\left\{\sum_{i+j=0}^{n-k} P_{i,j}(t)\right\}=\left\{\sum_{i+j=0}^{n-k}\frac{\partial}{\partial \lambda_1}P_{i,j}(t)\right\} \tag{38}$$

We use the same procedure to get $\frac{\partial R_N(k,n)}{\partial \lambda_2},\frac{\partial R_N(k,n)}{\partial \mu_1},\frac{R_N(k,n)}{\partial \mu_2}$.

We use two cases to study the effect of k and n on system reliability

Case 1: Fix $\lambda_1=0.001$, $\lambda_2=0.008, n=3$ and choose $k=1,2,3$.
Case 2: Fix $\lambda_1=0.001$, $\lambda_2=0.008, k=2$ and choose $n=2,3,4$.

From Fig.1-4 and Fig. 1-5 we can be observed that the system reliability increases as k increases or n increases.

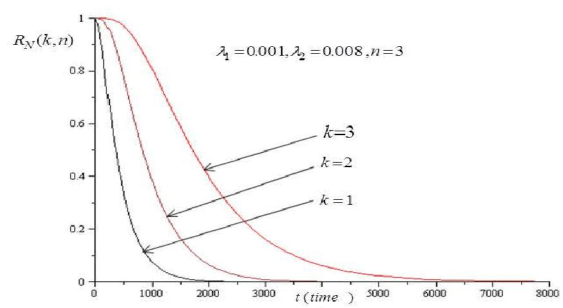

Figure1-4. Nonrepairable system reliability with two types of failure for different numbers of units k.

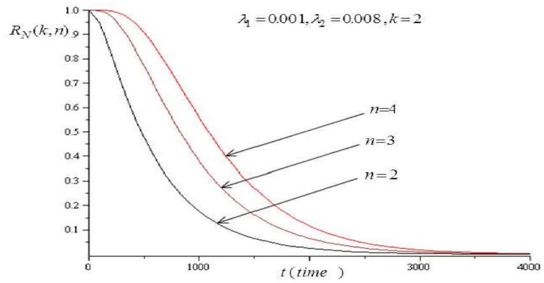

Figure1-5. Nonrepairable system reliability with two types of failure for different numbers of units n.

Then, we perform a sensitivity analysis with respect to λ_1 and λ_2. In Fig.1-6 we can easily observe that the biggest impact almost happened at different time and the order of magnitude of the effect is $(\lambda_1 > \lambda_1)$.

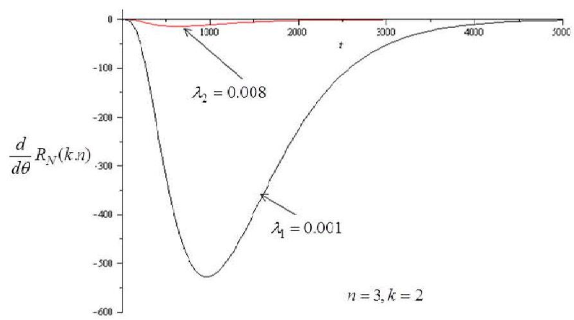

Figure1-6.sensitivity of repairable system reliability with respect to system parameters.

1.3 Repairable k-out-of-n System

In this section, we will develop a general model for analysis of such systems when they are repairable. Let failure rate and degradable rate that occur independently of the states of other units and follow exponential distributions with λ_1 and λ_2 respectively in addition, repair rates of failure and degradation are assumed to be exponentially distributed with parameters μ_1 and μ_2 respectively.

The system starts at time t=0 and there is no failed or degradable components. When a unit failed it is immediately sent to the first service line where it is repaired with time-to-repair which is exponentially distributed with parameter μ_1 .we have two service lines the first one repair failed units and the second one repair degraded units .When an operating unit degraded it is it is repaired with time-to-repair which is exponentially distributed with parameter μ_2 during it working. We assume that the secession of failure times and repair times are independently distributed random variable. Let us assume that failed units arriving at the repairmen form a single waiting line and are repaired in the order of their breakdowns; i.e. according to the first-come, first-served discipline. Suppose that the repairmen in the two service lines can repair only one failed unit at a time and the repair is independent of the failure of the units. Once a unit is repaired, it is as good as new.

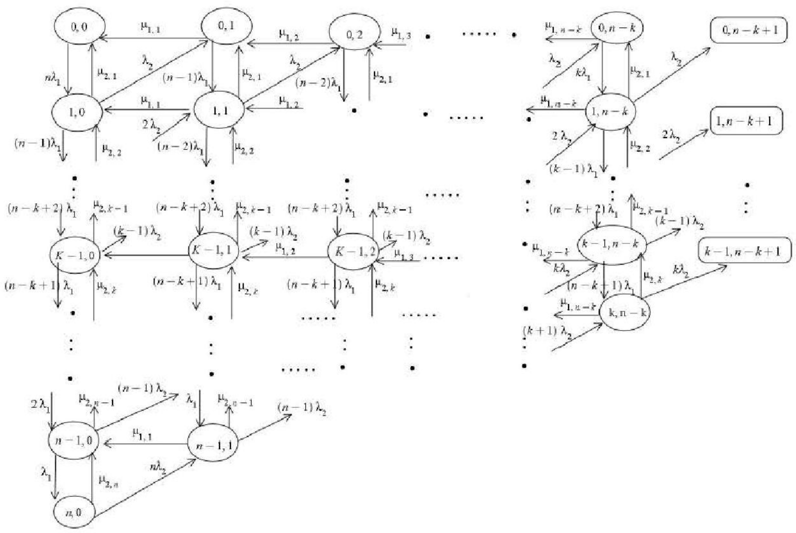

Figure1-7. State-transition-rate diagram of repairable System with two types of failure

The mean repair rate $\mu_{1,j}$ is given by:

$$\mu_{1,j} = \begin{cases} j\mu_1 & , \quad \text{if } 1 \le \text{n} \le \min(\text{R}_1, \text{n - k}) \\ R_1\mu_1 & , \quad \text{if } \text{R}_1 \le \text{j} \le \text{n} - k \\ 0 & \quad otherwise \end{cases}$$

The mean repair rate $\mu_{2,j}$ is given by:

$$\mu_{2,j} = \begin{cases} j\mu_2 & , \quad \text{if } 1 \le \text{n} \le \min(\text{R}_2, \text{n}) \\ R_2\mu_2 & , \quad \text{if } \text{R}_2 \le \text{j} \le \text{n} \\ 0 & \quad otherwise \end{cases}$$

Based on the model descriptions, the system state transition diagram is given in Fig.1-7 and it leads to the following Laplace transform expressions for $P_{i,j}^{*}(s)$:

$$(s + n\lambda_1)P_{0,0}^{*}(s) - \mu_{1,1}P_{0,1}^{*}(s) - \mu_{2,1}P_{1,0}^{*}(s) = 1 \quad (39)$$

$$(s + (n-j)\lambda_1 + \mu_{1,j})P_{0,j}^{*}(s) - \mu_{1,j+1}P_{0,j+1}^{*}(s) - \mu_{2,1}P_{1,j}^{*}(s) - \lambda_2 P_{1,j-1}^{*}(s) = 0$$
$$, 1 \le j \le n-k-1 \quad (40)$$

$$(s + k\lambda_1 + \mu_{1,n-k})P_{0,n-k}^{*}(s) - \lambda_2 P_{1,n-k-1}^{*}(s) = 0 \quad (41)$$

$$(s + (n-i)\lambda_1 + i\lambda_2 + \mu_{2,i})P_{i,0}^{*}(s) - (n-i+1)\lambda_1 P_{i-1,0}^{*}(s) - \mu_{1,1}P_{i,1}^{*}(s) - \mu_{2,i+1}P_{i+1,0}^{*}(s) = 0$$
$$, 1 \le i \le n-1 \quad (42)$$

$$\left(s+n\lambda_2+\mu_{2,n}\right)P^*_{n,0}(s)-\lambda_1 P^*_{n-1,0}(s)=0 \quad (43)$$

$$\left(s+(n-i)\lambda_2+\mu_{2,n-i}+\mu_{1,i}\right)P^*_{n-i,i}(s)-(n-i+1)\lambda_2 P^*_{n-i+1,i-1}(s)-\lambda_1 P^*_{n-i-1,i}(s)=0$$
$$,1\le i\le n-k \quad (44)$$

$$\left(s+i\lambda_2+(k-i)\lambda_1+\mu_{1,n-k}+\mu_{2,i}\right)P^*_{i,n-k}(s)-(i+1)\lambda_2 P^*_{i+1,n-k-1}(s)-\mu_{2,i+1}P^*_{i+1,n-k}(s)$$
$$-(k-i+1)\lambda_1 P^*_{i-1,n-k}(s)=0,1\le i\le k-1 \quad (45)$$

$$\left(s+(n-k-i)\lambda_2+\mu_{2,n-k-i}+\mu_{1,i}\right)P^*_{n-k-i,i}(s)-(n-k-i)\lambda_2 P^*_{n-k-i+1,i-1}(s)$$
$$-(n-k-1)\lambda_1 P^*_{n-k-i-1,i}(s)-\mu_{2,n-k-i+1}P^*_{n-k-i+1,i}(s)-\mu_{1,i+1}P^*_{n-k-i,i+1}(s)=0$$
$$,1\le i\le n-k-1 \quad (46)$$

$$\left(s+(n-k-j+i)\lambda_2+(k-i)\lambda_1+\mu_{1,j}+\mu_{2,n-k-j+i}\right)P^*_{n-k-j+i,j}(s)$$
$$-\mu_{2,n-k-j+i+1}P^*_{n-k-j+i+1,j}(s)-(n-k-j+i+1)\lambda_2 P^*_{n-k-j+i+1,j-1}(s)$$
$$-(k-i+1)\lambda_1 P^*_{n-k-j+i-1,j}(s)-\mu_{1,j+1}P^*_{n-k-j+i,j+1}(s)=0,1\le i\le k-1,1\le j\le n-k-1 \quad (47)$$

$$\left(s+(i+1)\lambda_2+(n-j-i-1)\lambda_1+\mu_{1,j}+\mu_{2,i+1}\right)P^*_{i+1,j}(s)-(n-j-i)\lambda_1 P^*_{i,j}(s)$$
$$-\mu_{2,i+2}P^*_{i+2,j}(s)-(i+2)\lambda_2 P^*_{i+2,j-1}(s)-\mu_{1,j+1}P^*_{i+1,j+1}(s)=0$$
$$,0\le i\le n-k-2,1\le j\le n-k-i-1 \quad (48)$$

$$(s)P^*_{i,n-k+1}(s)-(i+1)\lambda_2 P^*_{i+1,n-k}(s)=0,0\le i\le k-1 \quad (49)$$

By solving equations $(39-49)$ and taking inverse Laplace transforms (using maple program) .We obtain the reliability function as follows:

$$R_R(k,n)=L^{-1}\left(\sum_{i+j=0}^{n-k}P^*_{i,j}(s)\right)=\left(\sum_{i+j=0}^{n-k}P_{i,j}(t)\right) \quad (50)$$

Where $i,j=0,1,2,\ldots,n-k$.

The mean time to failure $MTTF_R(k,n)$ can be obtained from the following relation.

$$MTTF_R(k,n)=\lim_{s\to 0}\left\{\sum_{i+j=0}^{n-k}P^*_{i,j}(s)\right\}=\left\{\sum_{i+j=0}^{n-k}P^*_{i,j}(0)\right\} \quad (51)$$

We perform a sensitivity analysis for changes in the reliability of the system $R_R(k,n)$ from changes in system parameters $\lambda_1,\lambda_2,\mu_1$ and μ_2 .by differentiating equation (50) with respect to λ_1 we obtain

$$\frac{\partial R_R(k,n)}{\partial\lambda_1}=\frac{\partial}{\partial\lambda_1}\left\{\sum_{i+j=0}^{n-k}P_{i,j}(t)\right\}=\left\{\sum_{i+j=0}^{n-k}\frac{\partial}{\partial\lambda_1}P_{i,j}(t)\right\} \quad (52)$$

We use the same procedure to get $\frac{\partial R_R(k,n)}{\partial\lambda_2},\frac{\partial R_R(k,n)}{\partial\mu_1},\frac{R_R(k,n)}{\partial\mu_2}$.

1.3.1. Numerical Results

In this section, we use MAPLE computer program to provide the numerical results of the effects of various parameters on system reliability and system availability. We choose $\lambda_1=0.001$, $\lambda_2=0.008$ and fix $\mu_1=0.1,\mu_2=0.8$.The following cases are analyzed graphically to study the effect of various parameters on system reliability.

Case 1: Fix $n=5,k=2,R_2=1$, and choose $R_1=1,2,3$.

Case 2: Fix $n=5, k=2, R_1=1$, and choose $R_2=1,2,3$.

It can be observed from Fig.1-8 and Fig. 1-9 that the repairable system reliability increases as n or k increases. It is also noticed from Fig.1-10 that R_1 don't effect on system reliability when number of repairman more than one. Fig.1-11 shows that the repairable system reliability increases as R_2 increases.

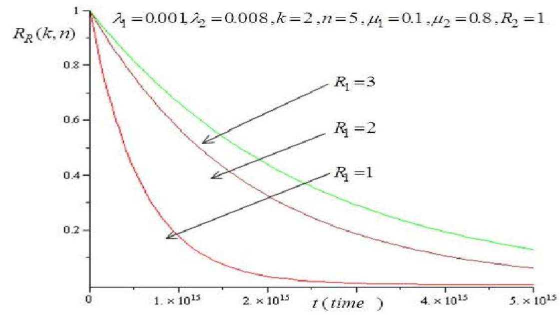

Figure1-8. Repairable system reliability for different numbers of repairmen in first service line.

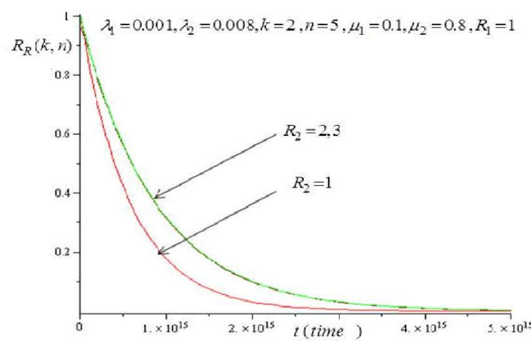

Figure1-9. Repairable system reliability for different numbers of repairmen in second service line.

Finally we perform sensitivity analysis for system reliability $R_R(k,n)$ with respect to system parameters.

1- With respect to all system parameters

From Fig.1-11 and Fig.1-12 we can easily observe that the biggest impact almost happened at the same time for λ_2, μ_1 and μ_2 but it's happen at shorter time for λ_1. Moreover, we find λ_1 is the most prominent parameter while λ_2, μ_1 and μ_2 are the second, the third and the fourth respectively in magnitude.

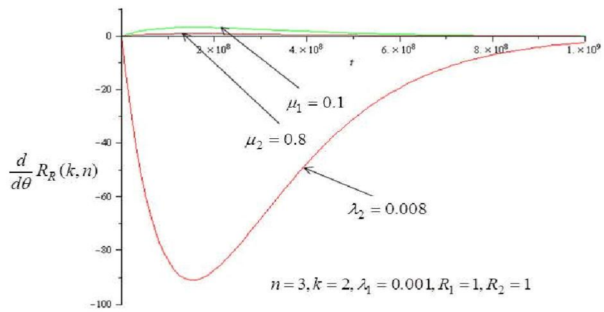

Figure 1-10. Sensitivity of system reliability with respect to system parameters.

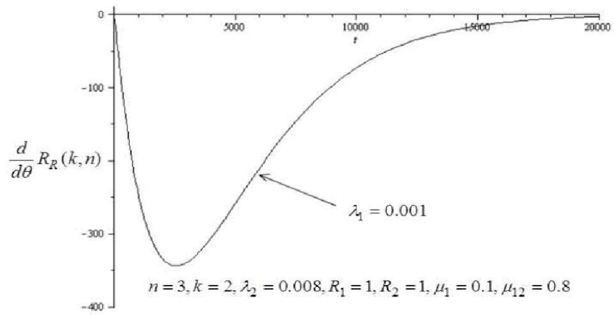

Figure 1-11. Sensitivity of system reliability with respect to system parameters.

2- With respect to λ_1 at various values of λ_1

Fig.2-6 shows that when λ_1 decreases the impact of λ_1 on reliability $R_Y(t)$ happened at longer interval time, and the biggest impact almost happened at longer time.

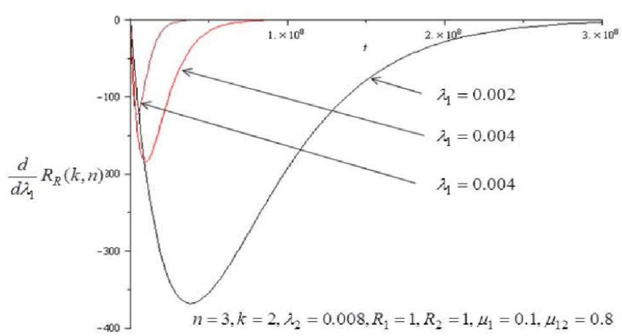

Figure 1-12. Sensitivity of system reliability with respect to λ_2 at various values of λ_1.

3- With respect to λ_1 at various values of λ_2

From Fig.2-7 we can also observe that as λ_2 decreases the impact of λ_1 on reliability $R_Y(t)$ happened at longer interval time, and the biggest impact almost happened at longer time, we can observe that the biggest impact of λ_1 on reliability $R_Y(t)$ is not affected by the value of λ_2 but it's happen at longer time as λ_2 decreases.

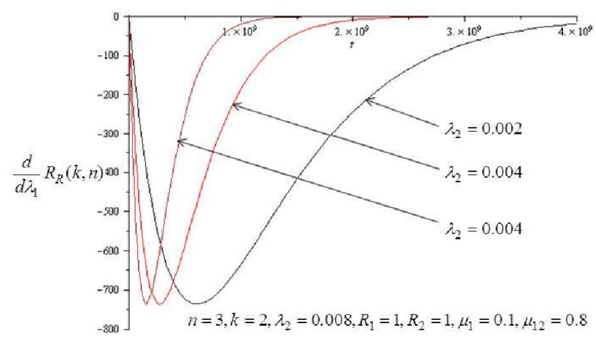

Figure 1-13. Sensitivity of system reliability with respect to λ_2 at various values of λ_2

1.4. Conclusions

In this paper, we studied reliability and mean time to system failure for k-out-of-n models with degradation facility. Mathematical model were constructed for these models. The results were shown graphically by the aid of MAPLE program.

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