Weibull-Halfnormal Mixture Distribution and Its Properties

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Abstract: This research study the generalization of Weibull and Half Normal Distribution (WHND) called Weibull Half Normal Distribution (WHND) through its distribution function and mathematical derivation of its moment, reliability, cumulative distribution function, and hazard rate function, probability density function. The distribution was found to generalize some known distributions thereby providing a great flexibility in modeling heavy tailed, skewed and bimodal distributions.
[Akomolafe. A.A, marad, A. And oladuti, O.M. **Weibull-Halfnormal Mixture Distribution and Its Properties.** *Rep Opinion* 2018;10(9):1-7]. ISSN 1553-9873 (print); ISSN 2375-7205 (online). http://www.sciencepub.net/report. 1. doi:10.7537/marsroj100918.01.

Keywords: Weibull-Halfnormal Distribution (WHND), Moment generating function, Hazard Function, Reliability, Hazard Function

1 Introduction

Weibull distribution is a continuous probability distribution. It is one of different distributions used to describe particle size with major application in survival analysis, weather forecast and reliability engineering. The half-normal distribution is a special case of the folded normal and truncated normal distributions. It was used to model brownian movement and can also be used in the modeling measurement data and lifetime data. Let X~N $(0,\sigma^2)$, then Y = |X| follows half normal distribution. The Half-normal is a fold at the mean of an ordinary normal distribution with mean zero, where σ is the scale parameter. Mixing weibull and halfnormal distribution together results to Weibull-halfnormal distribution. This proposed mixture distribution has more number of parameters as compared to their respective parent distributions and it has wider applicability exceeding modeling particle size but in modeling many stochastic processes and stochastic phenomena which cannot be easily modeled by two parameters probability density (parent distribution) such as disease growth, epidemiological studies of disease, buying behavior of consumers towards certain economic product etc.

1.1 Overview of the Research

Many researchers have worked on aspect of compounding two or more probability distributions to obtain family of hybrid distributions which are more efficient than their parent distributions due to addition of more parameters which increase the flexibility of the mixture of distributions in tracking many random phenomena which cannot be easily modeled by their parent distributions. Many authors have also worked on compounding beta distribution with other distributions. The beta family of distribution became popular some years back and which include beta-normal (Eugene & Famoye, 2002) [2]; beta-Gumbel (Nadarajah & Kotz, 2004) [3], beta-Weibull (Famoye, Lee & Olugbenga, 2005) [4], Beta-exponential (Nadarajah & Kotz, 2006) [5]: bata-Rayleigh (Akinsete & Lowe, 2009) [6]: beta-Laplace (Kozubowski Nadarajah, 2008) [7]: beta-pareto (Akinete, Faye & Lee, 200) [8], Barreto-Souza, Santos, and Cordeiro (2009) constructed Beta generalized exponential, Beta-half-Cauchy was presented by Cordeiro, and Lemonte (2011), Gastellares, Montenegero, and Gauss derived Beta Log–normal**,** while Morais, Cordeiro, and Audrey (2011) introduced Beta Generalized Logistic, the Beta Burr III Model for Lifetime Data, Beta–hyperbolic Secant (BHS) by Mattheas, David (2007), Beta Fre´chet by Nadarajah, and Gupta (2004), Beta-halfnormal by Akomolafe and Maradesa (2017), beta-Gamma, beta-f, beta-t, beta-beta, beta-modified weibull, beta-nakagami among others. Some articles also evolved regarding exponential-pareto by Kareema Abdul Al-Kadim and Mohammed Abdulhussain and exponential gamma mixture.

In the view of this, this research work aims at compounding Weibull with Halfnormal distribution so as to obtain its corresponding hybrid version called Weibull-Halfnormal Distribution (WHND). The proposed distributions can be used to describe many random phenomena.

2 Derivation of Weibull-Halfnormal Distribution (WHND)

The probability density function of the mixture of Weibull-Halfnormal distribution has the following form:

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$$f_{WHND}(x;\tau) = p_1 f_1(x;\tau_1) + p_2 f_2(x;\tau_2) \quad (1)$$

Where τ is the vector of parameters of WHND. τ_1 and τ_2 represent the parameters of the parent distributions. p_1 and p_2 are the mixing proportion, and $p_1 + p_2 = 1$.

where $f_1(x;\tau_1) = \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha}$ is the pdf of weibull distribution and $f_2(x;\tau_2) = \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}$

τ_1 and τ_2 are the parameter vector of weibull and halfnormal distribution respectively. From (1), the pdf of Weibull-Halfnormal can be developed as follows:

$$f_{WHND}(x;\tau) = p_1 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha} + p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} \quad (2)$$

$$p_1 + p_2 = 1 \text{ and } \alpha, \beta, \sigma > 0$$

Therefore, (2) is the pdf of Weibull-Halfnormal distribution (WHND). From (2), we can obtain the cdf of Weibull-Halfnormal distribution.

$$F_{WHND}(x;\alpha,\beta,\sigma) = \int_0^t f_{WHND}(x;\alpha,\beta,\sigma)\,dx + \int_0^t f_{WHND}(x;\alpha,\beta,\sigma)\,dx \quad (3)$$

$$F_{WHND}(x;\alpha,\beta,\sigma) = p_1\left(1 - e^{-(\frac{x}{\beta})^\alpha}\right) + p_2 \operatorname{erf}\left(\frac{x}{\sigma\sqrt{2}}\right) \quad (4)$$

The (4) is the cdf of Weibull-Halnormal distribution.

2 Moment

$$E(x^r) = \int_0^\infty x^r p_1 \frac{\alpha}{\beta^\alpha} x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha} dx + \int_0^\infty x^r p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}} dx \quad (5)$$

$$= p_1 \frac{\alpha}{\beta^\alpha} \int_0^\infty x^r x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha} dx + p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_0^\infty x^r e^{-\frac{x^2}{2\sigma^2}} dx \quad (6)$$

Let $y = \frac{x^2}{2\sigma^2}$; $\frac{dy}{dx} = \frac{x}{\sigma^2}$; $dx = \frac{\sigma^2}{x} dy$

$m = (\frac{x}{\beta})^\alpha$, $\frac{dm}{dx} = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}$, $dm = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} dx$, $dx = \frac{dm}{\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}}$

Put for dx, y and m in (6),

$$= p_1 \frac{\alpha}{\beta^\alpha} \int_0^\infty x^r x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha} \frac{dm}{\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}} + p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} \int_0^\infty x^r e^{-\frac{x^2}{2\sigma^2}} \frac{\sigma^2}{x} dy$$

$$= p_1 \frac{\alpha}{\beta^\alpha} \int_0^\infty x^r x^{\alpha-1} e^{-(\frac{x}{\beta})^\alpha} \frac{dm}{\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}} + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (7)$$

$$= p_1 \frac{\alpha}{\beta^\alpha} \cdot \frac{\beta}{\alpha} \int_0^\infty x^r x^{\alpha-1} e^{-m} \left(\frac{x}{\beta}\right)^{-(\alpha-1)} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (8)$$

$$= p_1 \beta^{1-\alpha} \int_0^\infty x^r x^{\alpha-1} e^{-m} \cdot \frac{x^{-(\alpha-1)}}{\beta^{-(\alpha-1)}} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (9)$$

$$= p_1 \beta^{1-\alpha} \int_0^\infty x^r e^{-m} \frac{1}{\beta^{-(\alpha-1)}} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (10)$$

$$= p_1 \beta^{1-\alpha} \beta^{(\alpha-1)} \cdot \int_0^\infty x^r e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (11)$$

Since $m = \left(\frac{x}{\beta}\right)^\alpha$; $m^{\frac{1}{\alpha}} = \frac{x}{\beta}$; $x = \beta m^{\frac{1}{\alpha}}$ and $y = \frac{x^2}{2\sigma^2}$; $x^2 = 2y\sigma^2$, $x = \sigma\sqrt{2y}$

Put for x in (12),

$$= p_1 \int_0^\infty x^r e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty x^{r-1} e^{-y} dy \quad (12)$$

$$= p_1 \int_0^\infty \left(\beta m^{\frac{1}{\alpha}}\right)^r e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty (\sigma\sqrt{2y})^{r-1} e^{-y} dy \quad (13)$$

$$= p_1\beta^r \int_0^\infty m^{\frac{r}{\alpha}} e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sigma^{r-1} (2y)^{\frac{1}{2}(r-1)} e^{-y} dy \quad (14)$$

$$= p_1\beta^r \int_0^\infty m^{\frac{r}{\alpha}} e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \int_0^\infty \sigma^{r-1} . 2^{\frac{r}{2}-\frac{1}{2}} . y^{\frac{r}{2}-\frac{1}{2}} e^{-y} dy \quad (15)$$

$$= p_1\beta^r \int_0^\infty m^{\frac{r}{\alpha}} e^{-m} dm + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \cdot \sigma^r \sigma^{-1} 2^{\frac{r}{2}} . 2^{-\frac{1}{2}} \int_0^\infty y^{\frac{r}{2}-\frac{1}{2}} . e^{-y} . dy \quad (16)$$

$$= p_1\beta^r \int_0^\infty m^{\frac{r}{\alpha}} e^{-m} dm + p_2 \frac{\sigma^r \sigma \sigma^{-1}}{\sqrt{\pi}} 2^{\frac{1}{2}} . 2^{-\frac{1}{2}} 2^{\frac{r}{2}} \int_0^\infty y^{\frac{r}{2}-\frac{1}{2}} . e^{-y} . dy \quad (17)$$

$$= p_1\beta^r \int_0^\infty m^{\frac{r}{\alpha}} e^{-m} dm + p_2 \frac{\sigma^r 2^{\frac{r}{2}}}{\sqrt{\pi}} \int_0^\infty y^{\frac{r}{2}-\frac{1}{2}} . e^{-y} . dy \quad (18)$$

$$= p_1\beta^r \Gamma\left(\frac{r}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^r}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{1}{2}\right) \quad (19)$$

$$Ex^r = p_1\beta^r \Gamma\left(\frac{r}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^r}{\sqrt{\pi}} \Gamma\left(\frac{r+1}{2}\right) \quad (20)$$

$$Ex = p_1\beta \Gamma\left(\frac{1}{\alpha}+1\right) + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}} \quad (21)$$

$$Ex^2 = p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^2}{\sqrt{\pi}} \Gamma\left(\frac{2+1}{2}\right) \quad (22)$$

$$Ex^2 = p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2 \frac{1.78\,\sigma^2}{\sqrt{\pi}} \quad (24)$$

$$Ex^3 = p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} \Gamma\left(\frac{3+1}{2}\right)$$

$$Ex^3 = p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} \quad (25)$$

$$Ex^4 = p_1\beta^4 \Gamma\left(\frac{4}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^4}{\sqrt{\pi}} \Gamma\left(\frac{4+1}{2}\right)$$

$$Ex^4 = p_1\beta^4 \Gamma\left(\frac{4}{\alpha}+1\right) + p_2 \frac{(\sigma\sqrt{2})^4}{\sqrt{\pi}} 1.3293$$

$$Ex^4 = p_1\beta^4 \Gamma\left(\frac{4}{\alpha}+1\right) + p_2 \frac{5.32\,\sigma^4}{\sqrt{\pi}} \quad (26)$$

The mean of Weibull-Halfnormal is, $Ex = p_1\beta \Gamma\left(\frac{1}{\alpha}+1\right) + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}}$

The variance is:

$$\text{Var}(x) = Ex^2 - (Ex)^2 = p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2 \frac{1.78\,\sigma^2}{\sqrt{\pi}} - \left(p_1\beta \Gamma\left(\frac{1}{\alpha}+1\right) + p_2 \frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2 \quad (27)$$

2.1 Skewness and Kurtosis

$$\mu_3 = E(x-\mu)^3$$

By applying Binomial Expansion

$$= E\left(\binom{3}{0} x^0 . (-\mu)^{3-0} + \binom{3}{1} x^1 . (-\mu)^{3-1} + \binom{3}{2} x^2 . (-\mu)^{3-2} + \binom{3}{3} . x^3 . (-\mu)^{3-3}\right)$$

$$= E((-\mu)^3 + 3x(-\mu)^2 + 3x^2(-\mu) + x^3) = E(x^3 - 3\mu x^2 + 3x\mu^2 - \mu^3)$$

$$= Ex^3 - 3\mu Ex^2 + 3\mu^2 Ex - \mu^3$$

$$\mu_3 = p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 3\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) . \left(p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) + 3\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) - \left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3 \quad (28)$$

$$= p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 3\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) . \left(p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) + 3\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3 - \left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3 \quad (29)$$

$$\mu_3 = p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 3\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) . \left(p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) + 2\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3 \quad (30)$$

$$\mu_4 = E(x-\mu)^4$$

$$= \binom{4}{0}.x^0.(-\mu)^{4-0} + \binom{4}{1}.x^1.(-\mu)^{4-1} + \binom{4}{2}x^2.(-\mu)^{4-2} + \binom{4}{3}.x^3.(-\mu)^{4-3} + \binom{4}{4}.x^4.(-\mu)^{4-4}$$

$$= E[(-\mu)^4 + 4x(-\mu)^3 + 6x^2(-\mu)^2 + 4x^3(-\mu) + x^4] =$$

$$= Ex^4 - 4\mu Ex^3 + 6\mu^2 Ex^2 - 4\mu^3 Ex + \mu^4$$

$$= p_1\beta^4 \Gamma\left(\frac{4}{\alpha}+1\right) + p_2\frac{5.32\,\sigma^4}{\sqrt{\pi}} - 4\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)\left(p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}}\right) + 6\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2.\left(p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) - 4\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3.\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right) + \left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^4$$

$$\mu_4 = p_1\beta^4 \Gamma\left(\frac{4}{\alpha}+1\right) + p_2\frac{5.32\,\sigma^4}{\sqrt{\pi}} - 4\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)\left(p_1\beta^3 \Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}}\right) + 6\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2.\left(p_1\beta^2 \Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) - 4\left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^4 + \left(p_1\beta\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^4 \quad (31)$$

$$\mu_4 = p_1\beta^4\,\Gamma\left(\frac{4}{\alpha}+1\right) + p_2\frac{5.32\,\sigma^4}{\sqrt{\pi}} - 4\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)\left(p_1\beta^3\,\Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}}\right) + 6\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2.\left(p_1\beta^2\,\Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) - 3\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^4$$

(32)

2.1.1 Skweness

$$\gamma_1(x) = \frac{(\mu_3)^2}{(\mu_2)^3} = \frac{\left(p_1\beta^3\,\Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}} - 3\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right).\left(p_1\beta^2\,\Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) + 2\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^3\right)^2}{\left(p_1\beta^2\,\Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}} - \left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2\right)^3}$$

2.1.2 Kurtosis

$$\gamma_2(x) = \frac{\mu_4}{(\mu_2)^2} = \frac{\left[p_1\beta^4\,\Gamma\left(\frac{4}{\alpha}+1\right) + p_2\frac{5.32\,\sigma^4}{\sqrt{\pi}} - 4\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)\left(p_1\beta^3\,\Gamma\left(\frac{3}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^3}{\sqrt{\pi}}\right) + 6\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2.\left(p_1\beta^2\,\Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}}\right) - 3\left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^4\right]}{\left(p_1\beta^2\,\Gamma\left(\frac{2}{\alpha}+1\right) + p_2\frac{1.78\,\sigma^2}{\sqrt{\pi}} - \left(p_1\beta\,\Gamma\left(\frac{1}{\alpha}+1\right) + p_2\frac{\sigma\sqrt{2}}{\sqrt{\pi}}\right)^2\right)^2}$$

2.2 Moment Generating Function

$$M_x(t) = Ee^{tx} = \int_0^\infty e^{tx} f(x;\tau)\,dx = \sum_j^\infty \frac{t^r}{r!}\int_0^\infty x^r f(x;\tau)\,dx = \sum_r^\infty \frac{t^r}{r!}\,Ex^r \qquad (33)$$

$$= \sum_j^\infty \frac{t^r}{r!}\,Ex^r = \sum_j^\infty \frac{t^r}{r!}\left(p_1\beta^r\,\Gamma\left(\frac{r}{\alpha}+1\right) + p_2\frac{(\sigma\sqrt{2})^r}{\sqrt{\pi}}\,\Gamma\left(\frac{r+1}{2}\right)\right) \qquad (34)$$

The (34) above is the mgf of Weibull-Halfnormal distribution

2.3 Reliability

$$R(x) = 1 - F_{WHND}(x;\alpha,\beta,\sigma) = 1 - \left[p_1\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right) + p_2\,\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right] \qquad (35)$$

2.4 Hazard Function

$$H(x) = \frac{f_{WHND}(x;\alpha,\beta,\sigma)}{R(t)} = \frac{p_1\frac{\alpha}{\beta^\alpha}x^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^\alpha} + p_2\frac{\sqrt{2}}{\sigma\sqrt{\pi}}e^{-\frac{x^2}{2\sigma^2}}}{1 - \left[p_1\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right) + p_2\,\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}$$

2.5 Odd function

$$\textbf{O (x)} = \frac{F(x;\alpha,\beta,\theta,\lambda)}{R(x)} = \frac{p_1\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right) + p_2\,\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)}{1 - \left[p_1\left(1 - e^{-\left(\frac{x}{\beta}\right)^\alpha}\right) + p_2\,\mathrm{erf}\left(\frac{x}{\sigma\sqrt{2}}\right)\right]}$$

2.6 Maximum Likelihood Estimation

$$Lf(x;\alpha,\beta,\sigma) = \prod_{i=1}^{n}\left(p_1 \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(\frac{x}{\beta})^{\alpha}} + p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\right) \quad (36)$$

The log likelihood function is given as:

$$\ln Lf(x;\alpha,\beta,\sigma) = \sum_{i=1}^{n}\ln\left(p_1 \frac{\alpha}{\beta^{\alpha}} x^{\alpha-1} e^{-(\frac{x}{\beta})^{\alpha}} + p_2 \frac{\sqrt{2}}{\sigma\sqrt{\pi}} e^{-\frac{x^2}{2\sigma^2}}\right) \quad (37)$$

By obtaining the partial derivatives with respect to each of the parameter and solve the resulting equation, we can obtain the estimate of the parameters which can be used to test the hypothesis about the consistency, stability and efficiency of the mixture distribution over its parent distributions. This can be done via numerical estimation, E-M algorithm in particular. The obtained information matrix can be employed in constructing confidence interval and testing hypothesis.

3 Conclusion

Compounding distributions lead to formulation of hybrid distribution with increased number of parameters as compared to its parent distributions. The proposed distribution is said to have more stability, consistency and flexibility in modeling data with heavily skewed or bimodal distribution.

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9/25/2018