The Hardy-Littlewood prime *k*-tuple conjecture is false

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Abstract: Using Jiang function we prove Jiang prime k -tuple theorem. We prove that the Hardy-Littlewood prime k -tuple conjecture is false. Jiang prime k -tuple theorem can replace the Hardy-Littlewood prime k -tuple conjecture.

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(A) Jiang prime k -tuple theorem [1, 2].

We define the prime k -tuple equation

$$p,\ p+n_i\ , \qquad (1)$$

where $2|n_i, i=1,\cdots k-1$.

we have Jiang function [1, 2]

$$J_2(\omega)=\prod_P(P-1-\chi(P)), \qquad (2)$$

where $\omega=\prod_P P$, $\chi(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1}(q+n_i)\equiv 0 \pmod P,\ q=1,\cdots,p-1. \qquad (3)$$

If $\chi(P)<P-1$ then $J_2(\omega)\neq 0$. There exist infinitely many primes P such that each of $P+n_i$ is prime. If $\chi(P)=P-1$ then $J_2(\omega)=0$. There exist finitely many primes P such that each of $P+n_i$ is prime. $J_2(\omega)$ is a subset of Euler function $\phi(\omega)$ [2].

If $J_2(\omega)\neq 0$, then we have the best asymptotic formula of the number of prime P [1, 2]

$$\pi_k(N,2)=\left|\{P\le N: P+n_i=prime\}\right|\sim\frac{J_2(\omega)\omega^{k-1}}{\phi^k(\omega)}\frac{N}{\log^k N}=C(k)\frac{N}{\log^k N} \qquad (4)$$

$$\phi(\omega)=\prod_P(P-1),$$

$$C(k)=\prod_P\left(1-\frac{1+\chi(P)}{P}\right)\left(1-\frac{1}{P}\right)^{-k} \qquad (5)$$

Example 1. Let $k=2, P, P+2$, twin primes theorem.

From (3) we have

$$\chi(2)=0,\quad \chi(P)=1 \text{ if } P>2, \qquad (6)$$

Substituting (6) into (2) we have

$$J_2(\omega)=\prod_{P\ge 3}(P-2)\neq 0 \qquad (7)$$

There exist infinitely many primes P such that $P+2$ is prime. Substituting (7) into (4) we have the best asymptotic formula

$$\pi_k(N,2)=\left|\{P\le N: P+2=prime\}\right|\sim 2\prod_{P\ge 3}(1-\frac{1}{(P-1)^2})\frac{N}{\log^2 N}. \qquad (8)$$

Example 2. Let $k=3, P, P+2, P+4$.
From (3) we have

$$\chi(2)=0, \quad \chi(3)=2 \tag{9}$$

From (2) we have

$$J_2(\omega)=0. \tag{10}$$

It has only a solution $P=3$, $P+2=5$, $P+4=7$. One of $P, P+2, P+4$ is always divisible by 3.
Example 3. Let $k=4, P, P+n$, where $n=2,6,8$.
From (3) we have

$$\chi(2)=0, \chi(3)=1, \chi(P)=3 \text{ if } P>3. \tag{11}$$

Substituting (11) into (2) we have

$$J_2(\omega)=\prod_{P\ge 5}(P-4)\ne 0, \tag{12}$$

There exist infinitely many primes P such that each of $P+n$ is prime.
Substituting (12) into (4) we have the best asymptotic formula

$$\pi_4(N,2)=\left|\{P\le N: P+n=prime\}\right| \sim \frac{27}{3}\prod_{P\ge 5}\frac{P^3(P-4)}{(P-1)^4}\frac{N}{\log^4 N} \tag{13}$$

Example 4. Let $k=5$, P, $P+n$, where $n=2,6,8,12$.
From (3) we have

$$\chi(2)=0, \chi(3)=1, \chi(5)=3, \chi(P)=4 \text{ if } P>5 \tag{14}$$

Substituting (14) into (2) we have

$$J_2(\omega)=\prod_{P\ge 7}(P-5)\ne 0 \tag{15}$$

There exist infinitely many primes P such that each of $P+n$ is prime. Substituting (15) into (4) we have the best asymptotic formula

$$\pi_5(N,2)=\left|\{P\le N: P+n=prime\}\right| \sim \frac{15^4}{2^{11}}\prod_{P\ge 7}\frac{(P-5)P^4}{(P-1)^5}\frac{N}{\log^5 N} \tag{16}$$

Example 5. Let $k=6$, P, $P+n$, where $n=2,6,8,12,14$.
From (3) and (2) we have

$$\chi(2)=0, \quad \chi(3)=1, \quad \chi(5)=4, \quad J_2(5)=0 \tag{17}$$

It has only a solution $P=5$, $P+2=7$, $P+6=11$, $P+8=13$, $P+12=17$, $P+14=19$. One of $P+n$ is always divisible by 5.

(B) The Hardy-Littlewood prime k-tuple conjecture[3-14].

This conjecture is generally believed to be true, but has not been proved(Odlyzko et al.1999).

We define the prime k-tuple equation

$$P, P+n_i \tag{18}$$

where $2|n_i, i=1,\cdots,k-1$.
In 1923 Hardy and Littlewood conjectured the asymptotic formula

$$\pi_k(N,2)=\left|\{P\le N: P+n_i=prime\}\right| \sim H(k)\frac{N}{\log^k N}, \tag{19}$$

where

$$H(k)=\prod_P\left(1-\frac{\nu(P)}{P}\right)\left(1-\frac{1}{P}\right)^{-k} \tag{20}$$

$\nu(P)$ is the number of solutions of congruence

$$\prod_{i=1}^{k-1}(q+n_i)\equiv 0 \pmod P, \quad q=1,\cdots,P. \tag{21}$$

From (21) we have $\nu(P)<P$ and $H(k)\neq 0$. For any prime k-tuple equation there exist infinitely many primes P such that each of $P+n_i$ is prime, which is false.

Conjecture 1. Let $k=2,\ P,\ P+2$, twin primes theorem

From (21) we have

$$\nu(P)=1 \tag{22}$$

Substituting (22) into (20) we have

$$H(2)=\prod_P \frac{P}{P-1} \tag{23}$$

Substituting (23) into (19) we have the asymptotic formula

$$\pi_2(N,2)=\left|\{P\le N: P+2=prime\}\right|\sim \prod_P \frac{P}{P-1}\frac{N}{\log^2 N} \tag{24}$$

which is false see example 1.

Conjecture 2. Let $k=3,\ P,\ P+2,\ P+4$.

From (21) we have

$$\nu(2)=1,\ \nu(P)=2 \text{ if } P>2 \tag{25}$$

Substituting (25) into (20) we have

$$H(3)=4\prod_{P\ge 3}\frac{P^2(P-2)}{(P-1)^3} \tag{26}$$

Substituting (26) into (19) we have asymptotic formula

$$\pi_3(N,2)=\left|\{P\le N: P+2=prime,\ P+4=prim\}\right|\sim 4\prod_{P\ge 3}\frac{P^2(P-2)}{(P-1)^3}\frac{N}{\log^3 N} \tag{27}$$

which is false see example 2.

Conjecture 3. Let $k=4$, $P,\ P+n$, where $n=2,6,8$.

From (21) we have

$$\nu(2)=1,\ \nu(3)=2,\ \nu(P)=3 \text{ if } P>3 \tag{28}$$

Substituting (28) into (20) we have

$$H(4)=\frac{27}{2}\prod_{P>3}\frac{P^3(P-3)}{(P-1)^4} \tag{29}$$

Substituting (29) into (19) we have asymptotic formula

$$\pi_4(N,2)=\left|\{P\le N: P+n=prime\}\right|\sim \frac{27}{2}\prod_{P>3}\frac{P^3(P-3)}{(P-1)^4}\frac{N}{\log^4 N} \tag{30}$$

Which is false see example 3.

Conjecture 4. Let $k=5$, $P,\ P+n$, where $n=2,6,8,12$

From (21) we have

$$\nu(2)=1,\ \nu(3)=2,\ \nu(5)=3,\ \nu(P)=4 \text{ if } P>5 \tag{31}$$

Substituting (31) into (20) we have

$$H(5)=\frac{15^4}{4^5}\prod_{P>5}\frac{P^4(P-4)}{(P-1)^5} \tag{32}$$

Substituting (32) into (19) we have asymptotic formula

$$\pi_5(N,2)=\left|\{P\le N: P+n=prime\}\right|\sim \frac{15^4}{4^5}\prod_{P>5}\frac{P^4(P-4)}{(P-1)^5}\frac{N}{\log^5 N} \tag{33}$$

Which is false see example 4.

Conjecture 5. Let $k=6$, P, $P+n$, where $n=2,6,8,12,14$.

From (21) we have

$$\nu(2)=1,\ \nu(3)=2,\ \nu(5)=4,\ \nu(P)=5 \text{ if } P>5 \qquad (34)$$

Substituting (34) into (20) we have

$$H(6)=\frac{15^5}{2^{13}}\prod_{P>5}\frac{(P-5)P^5}{(P-1)^6} \qquad (35)$$

Substituting (35) into (19) we have asymptotic formula

$$\pi_6(N,2)=\left|\{P\le N: P+n=prime\}\right|\sim\frac{15^5}{2^{13}}\prod_{P>5}\frac{(P-5)P^5}{(P-1)^6}\frac{N}{\log^6 N} \qquad (36)$$

which is false see example 5.

Conclusion. The Hardy-Littlewood prime k-tuple conjecture is false. The tool of additive prime number theory is basically the Hardy-Littlewood prime k-tuple conjecture. Jiang prime k-tuple theorem can replace Hardy-Littlewood prime k-tuple Conjecture. There cannot be really modern prime theory without Jiang function.

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