ON THE RESPONSE OF LOADED BEAM SUBJECTED TO MOVING MASSES AND EXTERNAL FORCES

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ABSTRACT: A theory describing the response of a loaded beam subjected to moving masses and external forces is considered. The governing equation is a fourth order partial differential equation. The finite Fourier sine transformation is used to transform the governing partial differential equation into second order ordinary differential equations. The ordinary differential equations for moving forces and moving masses are solved with Laplace transformation method. Number examples are used to demonstrate the efficiency of the solution. Numeric analysis shows that for a simple supported beam, the resonance frequency is lower with corresponding decrease in maximum amplitude when inertial is considered.
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Keyword: Beam, Transformation, Inertial, Resonance,
Classification: Dynamical System

1.0 INTRODUCTION

A beam or **"girder"** bridge is the simplest kind of bridge. In the past they may have taken the form of a log across a stream, but today they are more familiar to us as large box steel girder bridges. There are lots of different types of beam. A beam bridge needs to be stiff. It needs to resist twisting and bending under load. In its most basic form, a beam bridge consists of a horizontal beam that is supported at each end the pair. The weight of the beam pushes straight down on the pairs under load.

Moving loads causes solid bodies to vibrate intensively. Particularly at high velocities. Thus, the study of the behaviors of bodies subjected to moving lends has been the concern of several investigators. Among the earliest work in this area of study was the work of Timoshenko (1992) who considered the problem of simply supported Unite beams resting on an elastic Foundation and traversed by moving loads. In his analysis, he assumed that the loads were moving with constant velocities along the beam. Furthermore, Kenny (1954) took up the problem of investigating the dynamic response of infinite elastic beams on elastic foundation when the beam is under the influence of a dynamic load moving with constant speed. Lie included the effects of viscous damping in the governing differential equation of motion. More recently, Oni (1990) considered the problem of a harmonic time variable concentrated force moving at a uniform velocity over a Unite deep beam. The methods of integral transformations are used. In particular, the Unite Fourier transform is used for the length coordinate and the Laplace transform the time coordinate. Series solution, which converges as obtained for the deflection of simply supported beams. The analysis of the solution was carried out for various speeds of the load. Oni (2001) used the Galerkin method to obtain the response to several moving masses of a non-uniform beam resting on an elastic foundation. The effects of the elastic foundation on the transverse displacement of the non-uniform beam were analyzed for both the moving mass and the associated moving force problems. Awodola,T.O (2005) worked on the influence of foundation and axial force on the vibration of a simply supported thin (Bernoulli Euler) beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity is investigated in the paper. The governing equation is a fourth order partial differential equation. For the solution of this problem, in the first instance, the finite Fourier sine transformation is used to reduce the equation to a second order partial differential equation. The reduced equation is then solved using the Laplace transformation. Numerical analysis shows that the transverse deflection of the thin beam, resting on a uniform foundation, under the action of a variable magnitude harmonic load moving with variable velocity decreases as the foundation constant increases. It also shows that as the axial force increases, the transverse deflection of the thin beam decreases.

Furthermore, Milormir, Stanisic .M, and Hardin, J. C. (1969) developed a theory describing the response of a Bernoulli-Euler beam under an arbitrary number of concentrated moving masses. The theory is based on the Fourier technique and shows that, for a simply supported beam, the resonance frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered.

ii. Formulation of the problem

The vibration of a uniformly simply supported beam carrying an arbitrary number of discrete masses m, m_2, -----m_N is considered. This mass m_t is assumed to strike the beam at t = 0 and travel across it with velocity **v_it.** The equations of motion, with damping neglected is written as

$$\left[mA+\sum_{i=1}^{N}m_i\partial(u-v_it)\frac{\partial^{2y}}{\partial t^2}\right]+cd\frac{\partial y}{\partial t}+EI\frac{\partial^{4y}}{\partial x^4}+Ky+\frac{\rho\partial^{2y}}{\partial x^2}=gf(x,t) \quad (1)$$

Where f (x, t)= $\sum_{i=1}^{N}m_i\partial(x-v_it)$ (2)

EI = the flexural rigidity of the beam
f (x,t) = the transverse deflection of the beam
ρ = the mass density of the beam material
A = cross sectional area of the beam
g = the acceleration due to gravity
k = the foundation constant.
L = the length of the beam

$\partial(x-v_it)$ is the Dirac delta function define to be zero everywhere except x=v_it, i.e $\partial(x-v_it)$=0, x $\neq v_it$ and in addition $\int_0^L \partial(x-v_it)dx=1$ (3)

The boundary conditions are
Y (o, t) = Y (L, t) = 0
Y_{xx} (o, t) = Y_{xx} (L, t) = 0 (4)

Applying the Fourier finite sine transform, i.e

$$= \quad Z(m,t)=\int_0^L Y(x,t)Sin\frac{m\pi x}{L}dx \quad (5)$$

$$\int_0^L MA+\sum_{i=1}^{N}m_i\partial(x-v_it)\frac{\partial^{2y}}{\partial t^2}Sin\frac{m\pi x}{L}dx+cd\int_0^L\frac{\partial y}{\partial t}Sin\frac{m\pi x}{L}dx+EI\int_0^L\frac{\partial^{Ny}}{\partial x^4}Sin\frac{m\pi x}{L}dx$$

$$+Ky\int_0^L Y_{xx}Sin\frac{m\pi x}{L}dx+\rho\int_0^L\frac{\partial^{2y}}{\partial x^2}Sin\frac{m\pi x}{L}=\int_0^L F(x,t)Sin\frac{m\pi x}{L}dx \quad (6)$$

Solving equation (6) by integration by part, we obtained difference series solution which added together. The result becomes

$$EI\frac{m^4\pi^4}{L^4}Z(m,t)-p\frac{m^2\pi^2}{L^2}Z(m,t)+Kz(m,t)+MAZ_{tt}(m,t)+T\left\{\sum_{i=1}^{N}m_i\delta(x-v_it)\right\}Y_{tt}(m,t)$$

$$= g\sum_{i=1}^{N}m_iSin\frac{m\pi yit}{L} \quad (7)$$

Let $Q=EI\frac{m^4\pi^4}{L^4}-p\frac{m^2\pi^2}{L^2}+K$ (8)

$$Qz(m,t)+mAz_{tt}(m,t)+Cz_t(m,t)+T\left\{\sum_{i=1}^{N}m_i\delta(x-v_it)\right\}y_{tt}(m,t)$$

$$= \quad g\sum_{i=1}^{N}m_iSin\frac{m\pi yit}{L} \quad (9)$$

$$Z(m,t)+\frac{mA}{Q}Z_{tt}(m,t)+\frac{C}{Q}Z_t(m,t)+\frac{T}{Q}\left\{\sum_{i=1}^{N}m_i\delta(x-v_it)y_{tt}(m,t)\right\}$$

$$= \quad \frac{g}{Q}\sum_{i=1}^{N}m_iSin\frac{m\pi v_it}{L} \quad (10)$$

By the equation 9 we obtained the transformation equation.

$$Z_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=K_7\sum_{i=1}^{N}Sin\frac{m\pi v_i t}{L} \tag{11}$$

3.0 Solution of the transformed equation.

For the purpose of the solution we consider only one mass m traveling with velocity υ. The solutions for greater numbers of masses may be obtained in the same manner .Evidently the following special cases from equation 11 follow:

(a) Moving force: - If we neglect the inertia term, we have the classical case of a moving force. Under the above assumption equation 11 becomes

$$Z_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=K_7\sum_{i=1}^{N}Sin\frac{m\pi v_i t}{L} \tag{12}$$

(b) Moving Mass:-first approximation only the linear inertial term is considered, Equation 11 becomes.

$$Z_{tt}(m,t)+K_3\sum_{i=1}^{N}m_iZ_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=K_7\sum_{i=1}^{N}Sin\frac{m\pi v_i t}{L} \tag{13}$$

(a) For moving force

$$Z_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=K_7\sum_{i=1}^{N}m_iSin\frac{m\pi v_i t}{L}$$

Now solving equation for moving force, we have

$$Z_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=0$$

This implies that $Z(m,t)=Z_c+Z_p$ (14)

For Z_c; equation (12) becomes

$$Z(m,t)=e^{mt}$$

$$Z_t(m,t)=Me^{mt}$$

$$Z_{tt}(m,t)=M^2e^{mt}$$

$\Rightarrow$

$$M^2e^{mt}+K_5Me^{mt}=0$$

$$e^{mt}\left(M^2+K_5m+K_6\right)=0$$

$$e^{mt}\neq 0; then M^2+K_5m+K_6=0$$

By quadratic solution

$$M=\frac{-b\pm\sqrt{b^2-4ac}}{2a}\quad a=1,b=K_5 and c=K_6$$

$\Rightarrow$

$$M=\frac{-K_5\pm\sqrt{K_5^2-4K_6}}{2}$$

$$M_1=\frac{-K_5+\sqrt{K_5^2-4K_6}}{2},M_2=\frac{-K_5-\sqrt{K_5^2-4K_6}}{2}$$

Therefore

$$z_c=Ae^{m_1t}+Be^{m_2t}$$

$$=Ae^{\frac{\left(-K_5+\sqrt{K_5^2-4K_6}\right)t}{2}}+Be^{\frac{\left(-K_5-\sqrt{K_5^2-4K_6}\right)t}{2}}$$

Applying these boundary conditions
Z (m, o) = 1
Z_t (m, o) = 0
$\Rightarrow$

$$Ae^0 + Be^0 = 1 \quad or \quad A = 1 - B \tag{15}$$

$$\Rightarrow A + B = 1$$

for Z (m,o) = 0;

Let $K_7 = \left(\frac{-K_5 + \sqrt{K_5^2 - 4K_6}}{2} \right)^t$ and $K_8 = \left(\frac{-K_5 - \sqrt{K_5^2 - 4K_6}}{2} \right)^t$

$$\begin{aligned} Z(m,t) &= Ae^{K_7 t} + Be^{K_8 t} \\ Z_t(m,t) &= AK_7 e^{K_7 t} + BK_8 e^{K_8 t} \\ Z_t(m,o) &= AK_7 e^0 + BK_8 e^0 = 0 \\ &= AK_7 + BK_8 = 0 \end{aligned} \tag{16}$$

Sub (15) in (16)

$$(1 - B)K_7 + BK_8 = 0$$

$$K_7 - BK_7 + BK_8 = 0$$

$$BK_8 - BK_7 = -K_7$$

$$B(K_8 - K_7) = K_7$$

$$B = \frac{-K_7}{K_8 - K_7}$$

Therefore

$$A = 1 - \left(\frac{-K_7}{K_8 - K_7} \right)$$

$$= 1 + \frac{K_7}{K_8 - K_7} \quad \text{or} \quad \frac{K_8}{K_8 - K_7}$$

By expansion

$$A = \frac{K_5}{2\sqrt{K_5^2 - 4K_6}} + \frac{\sqrt{K_5^2 - 4K_6}}{2\sqrt{K_5^2 - 4K_6}}$$

$$A = \frac{K_5}{2\sqrt{K_5^2 - 4K_6}} + \frac{1}{2}$$

$$A = \frac{1}{2} + \frac{K_5}{2\sqrt{K_5^2 - 4K_6}}$$

$$B = 1 - A$$

$$\Rightarrow B = 1 - \tfrac{1}{2} + \frac{K_5}{2\sqrt{K_5^2 - 4K_6}}$$

$$= \frac{1}{2} + \frac{-K_5}{2\sqrt{K_5^2 - 4K_6}}$$

From $Z_c = Ae^{\left(-K_5+\sqrt{K_5^2-4K_6}\right)t/2} + Be^{\left(-K_5-\sqrt{K_5^2-4K_6}\right)t/2}$

i.e. $$Z_c = \left(\tfrac{1}{2} + \frac{K_5}{2\sqrt{K_5^2-4K_6}} e^{\left(-K_5+\sqrt{K_5^2-4K_6}\right)t/2} + \tfrac{1}{2} - \frac{K_5}{2\sqrt{K_5^2-4K_6}}\right) e^{\left(-K_5-\sqrt{K_5^2-4K_6}\right)t/2} \quad (17)$$

For Z_p;

$$Z_p = K_g \sin\frac{m\pi v_i t}{L}$$

Where $K_g = K_4 \sum_{i=1}^{N} mi$

$$Z_p = A\frac{\sin m\pi vt}{L} + B\frac{\cos m\pi vt}{L}$$

$$Z_{tp} = \frac{Am\pi vt}{L}\cos\frac{m\pi vt}{L} - \frac{Bm\pi vt}{L} Sinm\pi vt$$

$$Z_{ttp} = \frac{Am^2\pi^2v^2t^2}{L^2} Sin\frac{m\pi vt}{L} - \frac{Bm^2\pi^2v^2t^2}{L^2}\cos m\pi vt$$

$$\Rightarrow \frac{Am^2\pi^2v^2t^2}{L^2} Sin\frac{m\pi vt}{L} - \frac{Bm^2\pi^2v^2t^2}{L^2}\cos m\pi vt + K_5\left(\frac{Am\pi vt}{L}\cos\frac{m\pi vt}{L} - \frac{Bm\pi vt}{L} Sinm\pi vt\right.$$

$$\left. + K_6\left(A\sin\frac{m\pi vt}{L} + B\cos\frac{m\pi vt}{L} = K_9 Sin\frac{m\pi vt}{L}\right)\right.$$

$$Z = K_9\frac{\sin m\pi vt}{L}$$

$$Z_p = A\frac{\sin m\pi vt}{L} + B\frac{\cos m\pi vt}{L}$$

$$Z_{tp} = A\frac{m\pi vt}{L}\cos\frac{m\pi vt}{L} - B\frac{m\pi vt}{L} Sin\frac{m\pi vt}{L}$$

$$Z_{ttp} = -\frac{Am^2\pi^2v^2t^2}{L^2} Sin\frac{m\pi vt}{L} - \frac{Bm^2\pi^2v^2t^2}{L^2}\cos\frac{m\pi vt}{L}$$

The equation becomes

$$-\frac{Am^2\pi^2v^2t^2}{L^2} Sin\frac{m\pi vt}{L} - \frac{Bm^2\pi^2v^2t^2}{L^2}\cos\frac{m\pi vt}{L} + K_5\left(\frac{Am\pi vt}{L}\cos\frac{m\pi vt}{L} - \frac{Bm\pi vt}{L} Sin\frac{m\pi vt}{L}\right)$$

$$+ K_6\left(A\sin\frac{m\pi vt}{L} + B\cos\frac{m\pi vt}{L} = K_9 Sin\frac{m\pi vt}{L}\right)$$

$$=\left(-\frac{Am^2\pi^2v^2t^2}{L^2}-\frac{BK_5m\pi vt}{L}-Sin\frac{m\pi vt}{L}+AK_6Sin\frac{m\pi vt}{L}\right)$$

$$Sin\frac{m\pi vt}{L}+\left(K_5\frac{Am\pi vt}{L}-\frac{Bm^2\pi^2v^2t^2}{L^2}+BK_6\right) \quad (18)$$

$$\cos\frac{m\pi vt}{L}+K_9Sin\frac{m\pi vt}{L}$$

For, the solution Z (m,t) = $Z_c + Z_p$

$$Z(m,t)=\left(\tfrac{1}{2}+\frac{K_5}{2\sqrt{K_5^2-4K_6}}\right)\frac{1}{x-\frac{\left(-K_5+\sqrt{K_5^2-4K_6}\right)}{2}}+\left(\tfrac{1}{2}-\frac{K_5}{2\sqrt{K_5^2-4K_6}}\right)$$

$$\frac{1}{x-\frac{\left(-K_5-\sqrt{K_5^2-4K_6}\right)}{2}}+\left(\frac{K_9\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)^2+K_5^2-\frac{m^2\pi^2v^2t^2}{L^2}}\right)\frac{\frac{m\pi vt}{L}}{\left(\frac{m\pi vt}{L}\right)^2}+ \quad (20)$$

$$\left[\left(\frac{-K_9\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)^2+K_5^2-\frac{m^2\pi^2v^2t^2}{L^2}}\right)\frac{\frac{K_5m\pi vt}{L}}{K_6-\frac{m^2\pi^2v^2t^2}{L^2}}\right]\frac{x}{x^2+\left(\frac{m\pi vt}{L}\right)^2}$$

$$\mathrm{v}\,(\mathrm{x},\mathrm{t})=\left(\tfrac{1}{2}+\frac{K_5}{2\sqrt{K_5^2-4K_6}}\right)\frac{1}{S-\frac{\left(-K_5+\sqrt{K_5^2-4K_6}\right)}{2}}+\left(\tfrac{1}{2}-\frac{K_5}{2\sqrt{K_5^2-4K_6}}\right)$$

$$\frac{1}{S-\frac{\left(-K_5-\sqrt{K_5^2-4K_6}\right)}{2}}+\left(\frac{K_9\left(K_6-\frac{m\pi^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)^2+K_5-\frac{m^2\pi^2v^2t^2}{L^2}}\right)\left(\frac{m\pi vt}{-L^2S^2+m^2\pi^2v^2t^2}\right)+$$

$$\left[\left(\frac{-K_9\left(K_6-\frac{m^2\prod^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m\pi^2v^2t^2}{L^2}\right)^2+\frac{K_5^2m\pi^2v^2t^2}{L^2}}\right)\frac{\frac{K_5m\pi vt}{L}}{K_6-\frac{m^2\pi^2v^2t^2}{L^2}}\right]$$

((21)

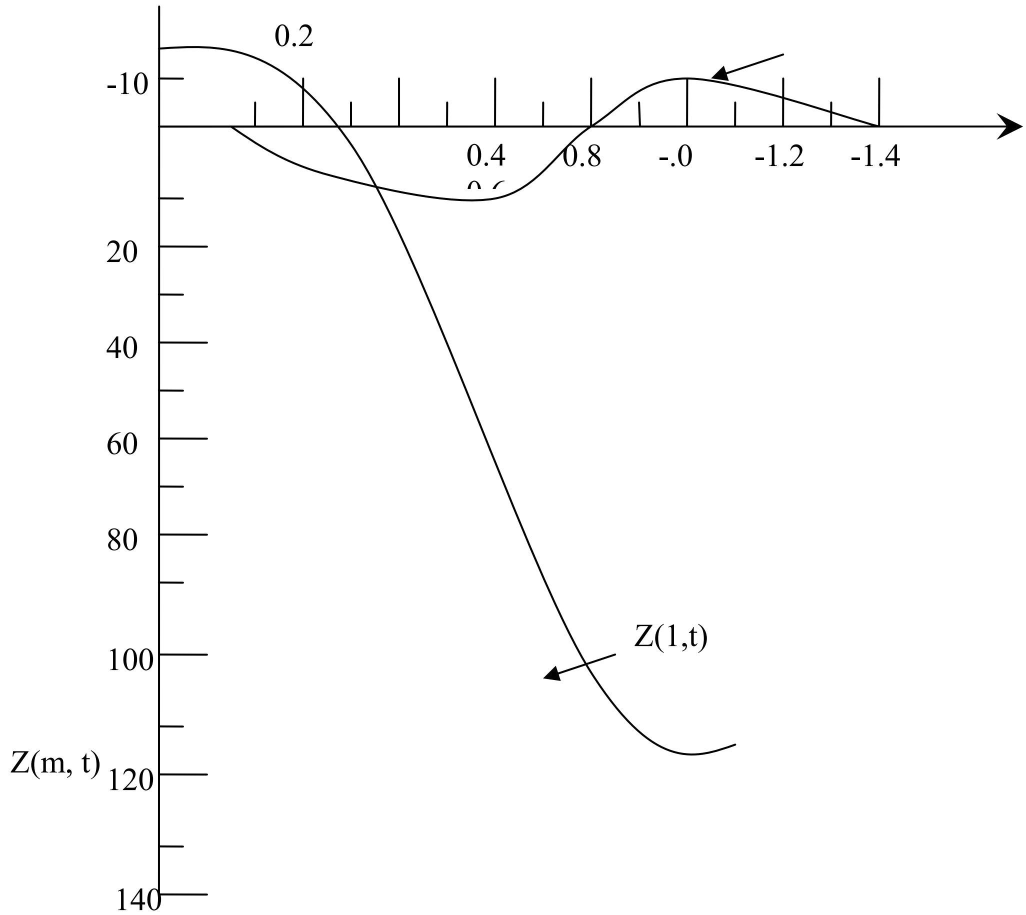

Fig. 2 Convergence of coefficient for moving mass solution

By using the same method for solving the equation for moving force i.e equation (13) for moving mass becomes:-

$$Z_{tt}(m,t)\left(1+K_3\sum_{i=1}^{N}Mi\right)+K_5Z_t(m,t)+K_6Z(m,t)=K_7(m,t)=K_7\sum_{i=1}^{N}Sin\frac{m\pi vt}{L}------(22)$$

And let $\left(1+K_3\sum_{i=1}^{N}Mi\right)=K_{10}$

$$K_{10}Z_{tt}(m,t)+K_5Z_t(m,t)+K_6Z(m,t)=K_9Sin\frac{m\pi vt}{L}$$

The solution for the moving mass becomes

$$V(m,t)=\left(\tfrac{1}{2}K_{10}+\frac{K_5}{2K_{10}\sqrt{K_5^2-4K_{10}K_6}}\right)\frac{1}{S-\frac{\left(-K_5+\sqrt{K_5^2-4K_{10}K_6}\right)}{2K_{10}}}+\left(\tfrac{1}{2}-\frac{K_5}{2K_{10}\sqrt{K_5^2-4K_{10}K_6}}\right)$$

$$\frac{1}{S-\frac{\left(-K_5-\sqrt{K_5^2-4K_{10}K_6}\right)}{2K_{10}}}+\left(\frac{K_9\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)^2+K_5\frac{m^2\pi^2v^2t^2}{L^2}}\right)\left(\frac{m\pi vt}{-L^2S^2+m^2\Delta^2v^2t^2}\right)+$$

$$\left[\left(\frac{-K_9\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)}{\left(K_6-\frac{m^2\pi^2v^2t^2}{L^2}\right)^2+\frac{K_5^2m^2\pi^2v^2t^2}{L^2}}\right)\frac{\frac{K_5m\pi vt}{L}}{K_6-\frac{m^2\pi^2v^2t^2}{L^2}}\right]------(23)$$

While this equation is resistant to analytic technique, it yields readily to numerical procedures. For z (m,t), the solutions for m=1,2 are tabulated in table below

Moving mass

S/N	t	Z(1,t)	Z(2,t)
1	0.0	0.000	0.000
2	0.1	0.139	0.300
3	0.2	1.078	1.726
4	0.3	3.479	4.581
5	0.4	7.785	8.025
6	0.5	14.209	9.009
7	0.6	22.766	6.817
8	0.7	33.311	1.686
9	0.8	45.559	3.321
10	0.9	59.089	7.369

Table 1.0 moving mass

Obviously higher approximations are possible by considering more terms of the series and following the same procedure. However, considering the rate of convergence of the lower-order solutions, it should not be necessary to continue this process.

4.0 Remark on the solution

In a problem such as this, one is interested in the maximum amplitude of vibration and the condition under which is can occur. For the classical solution, Eq.20, it can been shown that under certain values of the velocity the condition of resonance occurs. In this case, the amplitude of vibration becomes a linear function of time. Fig.5. the maximum value of the amplitude, which occurs when the mass is at the end of the beam.

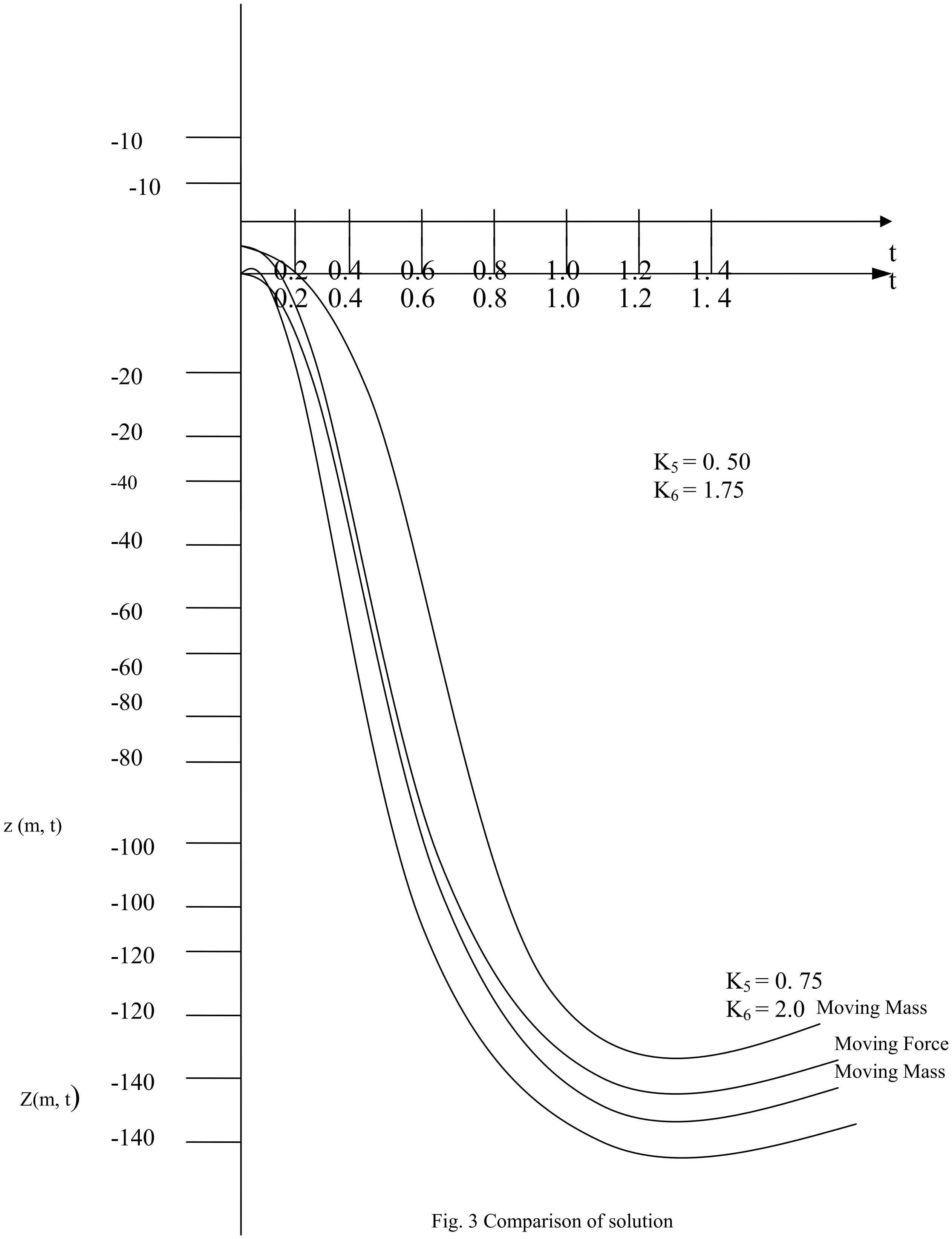

Fig. 3 Comparison of solution

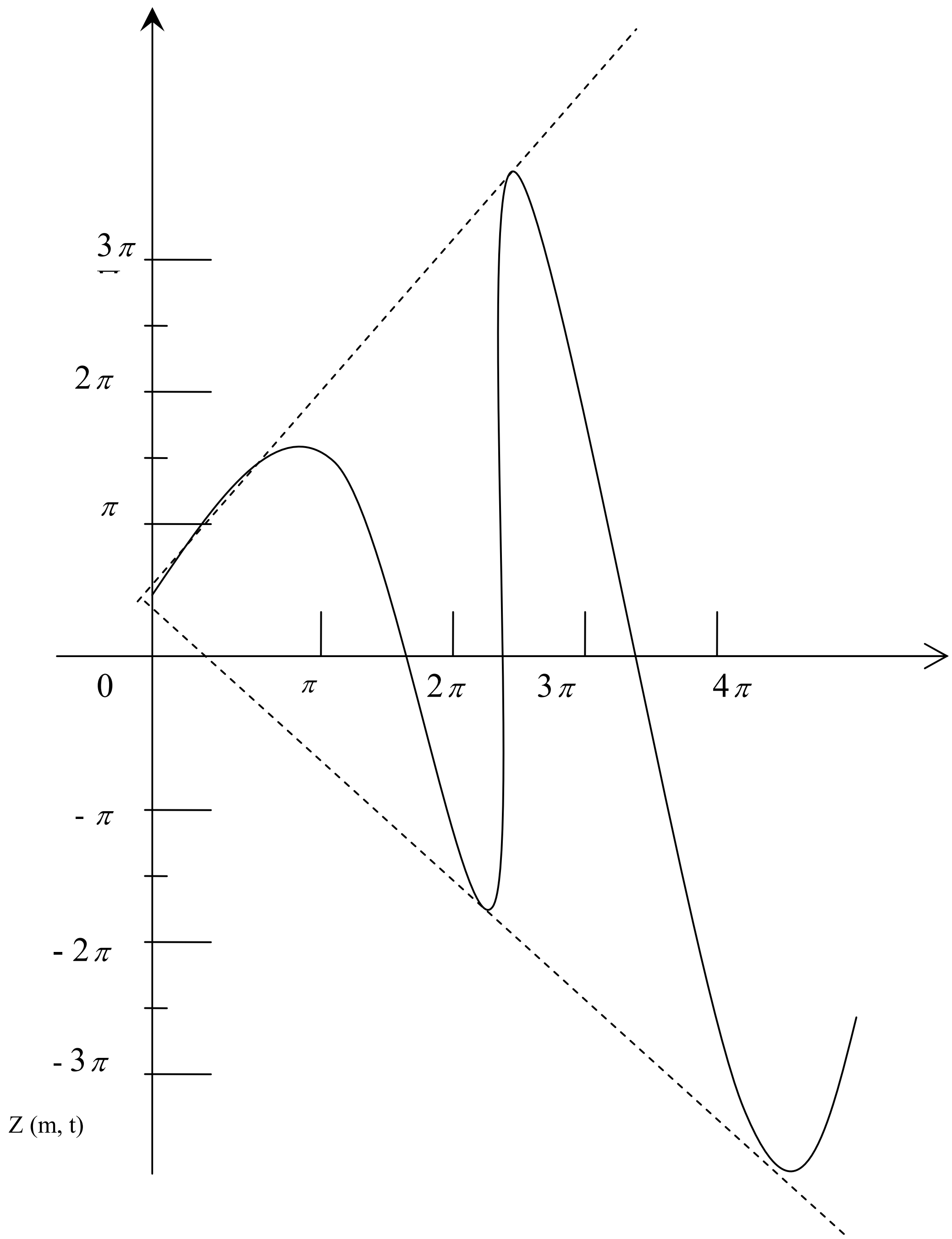

Fig. 5 Amplitude growth under resonant conditions

5.0 Conclusion

A theory is presented on the response of a loaded beam subjected to moving masses and external force. The theory is simple enough to be used in computation for design considerations. The equation of motion is given in terms of δ-Dirac functions and is solved through the use of Fourier finite sine transforms. An analytic approximation is obtained and compared with the solution for a moving force. The figure five shown the amplitude of the graph which growth under a resonance conditions and figure two shown the convergence of coefficient of moving mass solution, while figures 3 and 4 shows the comparison of the solution for moving mass and moving force, which moving mass is equal to moving force solution, of It is found that, for a simply supported beam, the resonant frequency is lower with no corresponding decrease in maximum amplitude when the inertia is considered. For any higher approximation, the solution can be obtained by means of numerical techniques and for future work; the convergence of the solution can be established.

6.0 RECOMMENDATION

(a) This work will assist the practicing engineer to evaluate the dynamic response of a loaded beam subjected to moving masses and external forces.

(b) This work can be applied to calculations involving prestressed or reinforced beams often encountered in structural design and Construction Company.

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