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Linear and quadratic algebraic equation and quadratic convergent iteration formulae

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Abstract: The problem of locating roots of nonlinear equations (or zeros of functions) occurs frequently in scientific work. In this paper, we have introduced some techniques for solving nonlinear equations. The techniques were based on the central-difference and forward-difference approximations to derivatives. We have shown that that three of the four methods have cubic convergence and another method has quadratic convergence. The introduced methods can be used for solving nonlinear equations without computing derivatives. Meanwhile, the methods introduced in this paper can be used to more class of nonlinear equations. The numerical examples shown in this paper illustrated the efficiency of the new methods. We used the well known software MATLAB 7 to calculate the numerical results obtained from the proposed techniques.
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1.1 Introduction

The relaxed Newton's method modifies the classical Newton's method with a parameter in such a way that when it is applied to a polynomial with multiple roots and we take as parameter one of these multiplicities, the order of convergence to the related multiple root is increased. For polynomials of degree three or higher, the relaxed Newton's method may possess extraneous attracting (or even super-attracting) cycles.[1] The eighth chapter presents some algorithms and implementations that allow us to compute the measure (area or probability) of the basin of a p-cycle when it is taken in the Riemann sphere. We quantify the efficiency of the relaxed Newton's method by computing, up to a given precision, the measure of the different attracting basins of non-repelling cycles. In this way, we can compare the measure of the basins of the ordinary fixed points (corresponding to the polynomial roots) with the measure of the basins of the point at infinity, and the basins of other non-repelling p-cyclic points for $p > 1$: The aim of the ninth chapter is to provide an overview of theoretical results and numerical tools in some iterative schemes to approximate solutions of nonlinear equations.[2] We examine the concept of iterative methods and their local order of convergence, numerical parameters that allow us to assess the order, and the development of inverse operators (derivative and divided differences). We also provide a detailed study of a new computational technique to analyze efficiency. Finally, we end the chapter with a discussion on adaptive arithmetic to accelerate computations.[3]

Most of the real life-problems are non-linear in nature therefore it is a challenging task for the mathematician and engineer to find the exact solution of such problems. In this reference, a number of methods have been proposed/implemented in the last two decades. Analytical solutions of such non-linear equations are very difficult, therefore only numerical method based iterative techniques are the way to find approximate solution. In the literature, there are some numerical methods such as *Bisection*, *Secant*, *Regula-Falsi*, *Newtonphson*, *Mullers* methods, etc., to calculate an approximate root of the non-linear transcendental equations. It is well known that all the iterative methods require one or more initial guesses for the initial approximations.[4]

In Regula-Falsi method, two initial guesses are taken in such a way that the corresponding function values have opposite signs. Then these two points are connected through the straight line and next approximation is the point where this line intersect the x-axis. This method gives guaranteed result but slow convergence therefore several researchers have improved this standard Regula-Falsi method into different hybrid models to speed up the convergence. Thus previously published works have revised/implemented Regula-Falsi method in several ways to obtain better convergence. However, it is found that modified form of Regual-Falsi method becomes more complicated from computational point of view.[5] Therefore, in the present work Regual-Falsi method has been used as its standard form with Newton–Raphson method and found better convergence. Newton–Raphson method is generally used to improve the result obtained by one of the above methods. This method uses the concept of *tangent* at the initial approximation point. The next approximate root is taken those value where the tangent intersect the x-axis. So this method fails where tangent is parallel to x-axis, i.e. the derivative of the function is zero or approximately zero. The order of convergence of Newton–Raphson method is two, therefore it converges very rapidly than other methods (Bisection, Regula-Falsi, etc.). However it does not always give guaranteed root. Many scientists and engineers have been proposed different hybrid models on Newton–Raphson method.[6]

It is clear from the survey, that the most of new algorithms are either based on three classical methods namely Bisection, Regula-Falsi and Newton–Raphson or created by hybrid processes. In the present work, the proposed new algorithm is based on standard Regula-Falsi and Newton–Raphson methods, which provides guaranteed results and higher order convergence over Regula-Falsi method. The new proposed algorithm will work even the first derivative equals to zero where Newton–Raphson method fails.[7]

A large number of papers have been written about iterative methods for the solution of the nonlinear equations [3, 7, 8, 9, 10, 12, 13]. In this paper, we consider the problem of finding a simple root x∗ of a function f : D ⊂ R → R i.e., f (x∗) = 0 and f 0 (x∗) 6= 0. The famous Newton's method for finding x∗ uses the iterative method:

xn+1 = xn − f (xn)

f **0** (xn) ,

REplace '

Newton's Raphson method is a very simple and elegant technique to find out the roots of a wide variety of the problems. But it has a drawback that it may fail if the derivative is approaching to zero near the root or the initial guess is not proper. In this work an alternative to Newton's method is presented by the authors.[8]

One of the most studied problems in Numerical Analysis is the approximation of nonlinear equations. A powerful tool is the use of iterative methods. It is well-known that Newton's method,

Most used iterative methods to approximate the solution x of F.x/ D 0. The quadratic convergence and the low operational cost of Newton's method ensure that it has a good computational efficiency. If we are interesting in methods without using derivatives, then Steffensen-type methods will be a good alternative. These methods only compute divided differences.[9]

1.2 A Family of New Algorithms

Two different classes of iteration techniques to find the roots are presented.[10]

1.2.1 Consider the equation $f(x) = 0$ whose roots are to be found. Let α be the exact root and x_0 be the initial guess known for the required root. Assume the first approximation to the required root as $x_1 = x_0 + h$, where h is very small.

1.2.1(a) Consider the following auxiliary equation with a parameter p

$$g(x) = p^2(x-x_0)^2 f^2(x) - f^2(x) = 0 \qquad \ldots(1.1)$$

where $p \in R$ and $|p| < \infty$. The root of $f(x)$ will also be a root of equation (1.1) and vice versa. Since $x = x_1 = x_0 + h$ is an approximation of the required root, therefore equation (1.1) gives,

$$p^2 h^2 f^2(x_0+h) - f^2(x_0+h) = 0,$$

Expanding by Taylor's theorem (retaining the terms up to $O(h^2)$ and excluding the term containing second derivative)

$$p^2h^2\left[f(x_0)+hf'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots\right]^2 - \left[f(x_0)+h\,f'(x_0)+\frac{h^2}{2!}f''(x_0)+\ldots\right]^2 = 0$$

$$\Rightarrow \quad p^2h^2\left[f^2(x_0)+h^2f'^2(x_0)+2h\,f(x_0)f'(x_0)+\ldots\right]$$
$$-[f^2(x_0)+h^2f'^2(x_0)+2h\,f(x_0)f'(x_0)+\ldots]=0,$$

$$\Rightarrow \quad h^2[p^2f^2(x_0)-f'^2(x_0)]-2hf(x_0)f'(x_0)-f^2(x_0)=0,$$

$$\Rightarrow \quad h=\frac{2f(x_0)f'(x_0)\pm\sqrt{4f^2(x_0)f'^2(x_0)+4f^2(x_0)[p^2f^2(x_0)-f'^2(x_0)]}}{2[p^2f^2(x_0)-f'^2(x_0)]},$$

$$\Rightarrow \quad h=\frac{f(x_0)f'(x_0)\pm\sqrt{f^2(x_0)[p^2f^2(x_0)]}}{p^2f^2(x_0)-f'^2(x_0)},$$

$$\Rightarrow \quad h=\frac{f(x_0)f'(x_0)\pm pf^2(x_0)}{p^2f^2(x_0)-f'^2(x_0)}$$

To avoid the loss of significant errors implicit in this formula, numerator is rationalized to obtain the formula,

$$h=\frac{f(x_0)[f'(x_0)\pm pf(x_0)]}{[p^2f^2(x_0)-f'^2(x_0)]}\times\frac{[f'(x_0)\mp pf(x_0)]}{[f'(x_0)\mp pf(x_0)]},$$

$$\Rightarrow \quad h=\frac{f(x_0)[f'^2(x_0)-p^2f^2(x_0)]}{[p^2f^2(x_0)-f'^2(x_0)]}\times\frac{1}{[f'(x_0)\mp pf(x_0)]},$$

$$\Rightarrow \quad h=\frac{-f(x_0)}{f'(x_0)\pm pf(x_0)}, \qquad \text{...(1.2)}$$

In this the sign is so chosen to make the denominator largest in magnitude. The first approximation to the required root is given by,

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)\pm pf(x_0)},$$

Therefore, the successive approximations are given by,

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)\pm pf(x_n)}, \quad n=0,1\ldots \qquad \text{...(1.3)}$$

The parameter p is chosen such that the corresponding function $pf(x_n)$ and $f'(x_n)$ have the same signs, so as to make the denominator largest possible. Letting $p\to 0$ in equation (1.3), reduces to Newton's formula.

1.2.1 (b) When an auxiliary equation of the following form is assumed,

$$g(x)=p^2(x-x_0)^2f(x)-f(x)=0 \qquad \text{...(1.4)}$$

where $p\in R$, the root of the equation $f(x)=0$ is also the root of equation (1.4). Putting $x=x_1=x_0+h$ in equation (1.4) and proceeding as in section 1.2.1 (a) general formula for successive approximation is given by,

$$x_{n+1}=x_n-\frac{2f(x_n)}{f'(x_n)\pm\sqrt{f'^2(x_n)+4p^2f^2(x_n)}}, \quad n=0,1\ldots \qquad \text{...(1.5)}$$

The sign should be so chosen so as to the denominator is largest in magnitude. Again equation (1.5), reduces to Newton's formula, if $p\to 0$. [11]

1.3 Convergence Analysis

Let α is a root of $f(x)=0$. An approximation of the root is given by $x_n=\alpha+e_n$, where e_n is error.

Using Taylor's series, expansions of $f(x_n)$ and $f'(x_n)$ given by equations (2.9) and (2.10), one gets, [12]

$$\frac{f(x_n)}{f'(x_n)}=e_n-C_2e_n^2+2(C_2^2-C_3)e_n^3+O(e_n^4), \qquad \text{...(1.6)}$$

Also

$$\left[\frac{f(x_n)}{f'(x_n)}\right]^2=e_n^2+O(e_n^3) \qquad \text{...(1.7)}$$

Equations (1.5), can be rewritten as,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)\left[1 + p^2 \dfrac{f^2(x_n)}{f'^2(x_n)}\right]} \quad \ldots(1.8)$$

Using equations (1.6) and (1.7) in equations (1.8), one gets,

$$e_{n+1} = (C_2 + p^2)e_n^2 + O(e_n^3).$$

which shows that the technique is quadratically convergent for each $p \in R$. Similarly, sequence $\{x_n\}$ generated by iteration formula (1.3), with parameter p, can be proved to be at least quadratically convergent.

Table 1.1 : Comparison of equation (1.5) with Newton's method.

S.N.	Equation	Initial Guess	Newton's Method	Equation (1.5) for p=1
1.	$x^{10} - 1 = 0$	0.0	Fails	1
2.	$x^2 - 4 = 0$	0.0	Fails	2
3.	$4x^4 - 4x^2 = 0$	$\frac{\sqrt{21}}{7}$	Divergent	0
4.	$\tan^{-1} x = 0$	3	Divergent	0
1.	$\sin x = 0$	1.5	-12.5663709641	0
1.	$\ln x = 0$	5	Divergent	1
7.	$e^{x^2+7x-30} - 1 = 0$	2	Divergent	3
8.	$e^x - 1 - \cos \pi x = 0$	-0.10	-7.3182411194	0.3692564070

1.4 Numerical Examples

A comparison of the formula proposed in section 1.2.1(b) with Newton's method is presented in Table 1.1 with the help of various examples. The formulae of section 1.2.1(b) are tested for p=1. The termination criterion is taken as $|f(x)| < 1.0 \times 10^{-11}$. [13,14]

We used the well known software MATLAB 7 to calculate the numerical results obtained from the proposed techniques.

From Table 1.1, it is observed that the formula (1.5) can be used as an alternative to Newton's technique, for the problems for which the latter fails or diverges.

1.5 Conclusions

The problem of locating roots of nonlinear equations (or zeros of functions) occurs frequently in scientific work. In this paper, we have introduced some techniques for solving nonlinear equations. The techniques were based on the central-difference and forward-difference approximations to derivatives. We have shown that that three of the four methods have cubic convergence and another method has quadratic convergence. The introduced methods can be used for solving nonlinear equations without computing derivatives. Meanwhile, the methods introduced in this paper can be used to more class of nonlinear equations. The numerical examples shown in this paper illustrated the efficiency of the new methods.

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