New York Science Journal

Websites:
http://www.sciencepub.net/newyork
http://www.sciencepub.net

Emails:
newyorksci@gmail.com
editor@sciencepub.net

MARSLAND PRESS
Multidisciplinary Academic Journal Publisher

An Analytical Approach to Pricing Discrete Barrier Options under Time-dependent Models

Mohammad H. Beheshti[1], Amir T. Payandeh Najafabadi[2] and Rahman Farnoosh[3]

[1] Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
[2] Faculty member of Department of Mathematical Sciences, Shahid Beheshti University, Tehran, Iran.
[3] School of Mathematics, Iran University of science and technology, Tehran, Iran.
mohammadhoseinbeheshti@yahoo.com

Abstract: Consider the problem of pricing a discrete barrier option under a time-dependent framework. This article provides an analytical solution for such an interesting problem in two steps. Namely, in the first step, the problem in hand restates a time invariant which has an exact solution. Secondly, the exact solution for the time-dependent model arrives by substituting such a solution in an integral equation. Applications to the Greeks of the contracts are given.

[Mohammad H. Beheshti, Amir T. Payandeh Najafabadi and Rahman Farnoosh. **An Analytical Approach to Pricing Discrete Barrier Options under Time-dependent Models.** *N Y Sci J* 2020;13(8):63-71]. ISSN 1554-0200 (print); ISSN 2375-723X (online). http://www.sciencepub.net/newyork. 7. doi:10.7537/marsnys130820.07.

Keywords: Barrier option, Black-Scholes framework, Discrete monitoring, Time-dependent model.

1. Introduction

The problem of pricing a discrete barrier option plays a role in the quantitative finance and financial industry. Barrier options are usually traded as the modification of simple European puts and call options. Barrier options activated (knock-ins) or terminated (knock-outs) if the sample path of the underlying asset has crossed a predetermined barrier prior to the exercise time. There are several pricing formulaes for barrier options in the Black-Scholes framework (see, Plesser 2000; Geman & Yor 1996; Rich 1994). In practice, barrier options differ from those studied in the academic literature in many respects. One of the most important is the monitoring frequency of the underlying assets. In the case of discrete monitoring, the sample path of the underlying asset is monitored at fixed times. Heynen & Kat (1995) were probably the scholars who issued an article noticing the discrepancy between option price under continuous and discrete monitoring. After seminal work of Heynen & Kat (1995), several authors proposed approximations based on a variety of different numerical approaches (see, Ait-sahlia & Lai, 1997; Bertoldi & Bianchetti, 2003; and Broadie & Glasserman, 1997). They used several methods such as: Recursive integration method, monte carlosimulation and trinomial tree. In the numerical approaches, computational cost increases whenever the number of monitoring increases. Moreover, the accuracy of the approximated solution decreases whenever asset price and the barrier are close to each other, in some sense. To overcome such difficulties, Fusai et al. (2006) provided an analytical method for the problem of pricing a discrete barrier option under time invariant framework.

For the problem of pricing a discrete barrier option under time-dependent framework (time-dependent parameters) the above findings are not valid anymore. More precisely, the problem of pricing a barrier option with time-dependent parameters is not a trivial extension of time invariant model. Roberts & Shorthland (1997) applied the hazard rate tangent approximation method to evaluate upper and lower bounds for price of such time-dependent barrier option. Unfortunately, their bounds cannot state in the closed form and consequently cannot be improved. Lo et al. (2003) presented a simple approach for computing upper and lower bounds (in the close form) for the price of a barrier option.

This article considers the problem of pricing a down-and-out discrete barrier options on a divided paying equity whenever the risk-free rate and the dividend yield in the Black-Scholes partial differessential equation are deterministic functions of time. A method of reducing time-dependent partial differential equation to the heat equation is described in Wilmott et al. (1999). Also Marianito & Rogemar (2006) outlined a procedure that transforms the Black-Scholes partial differential equation with time-dependant parameters into the Black-Scholes equation with time-independent parameters. Our approach is to reduce the pricing problem to the time-independent case and the solution to the latter equation provided by Fusai et al. (2006).

Section 2 collects some useful elements for the other sections. Section 3 provides the price of discrete barrier option under the time-dependent parameters. The price of the Greeks of contract along with two examples are given in Section 3.

2. Material and Methods

Now, we collect some essential elements for the next sections. Suppose $0 = t_0 < t_1 < ... < t_N = T$ be the monitoring dates that take at necessarily equally-spaced points in time, T the option maturity, the L the constant lower barrier which is active at all times tn and K is the strick price of the option. The nth time interval is defined as $t_n < t < t_{n+1}$. We are interested in pricing a down-and-out call option, i.e., a call option that expires worthless if a lower barrier has been hit at monitoring date. We denote $V(S, t, n)$ is the price of a down and-out barrier option on a dividend-paying equity at time t in the nth time interval and the asset price S. The asset price S satisfies the following stochastic differential equation.

$$\frac{dS}{S}=(r(t)-d(t))+\sigma dB$$

with the constant volatility σ, the risk-free rate $r(t)$ and the dividend yield $d(t)$, then it is well known (see, Merton (1973)) that $V(S, t, n)$ satisfies following partial differential equation:

$$\frac{\partial V(S, t, n)}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 V(S, t, n)}{\partial S^2}+\left[r(t)-d(t)\right]S\frac{\partial V(S, t, n)}{\partial S}-r(t)V(S, t, n)=0 \tag{1}$$

Given that the trigger condition is checked only at fixed times, it is needed to update the initial condition at each of the monitoring dates t_n:

$$V(S, t_n, n) = V(S, t_n, n-1)I_{(S\geq L)}, \tag{2}$$

$$V(S, t_0, 0) = (S-K)I_{(S\geq \max\{K,L\})}. \tag{3}$$

Now, suppose $\bar{V}(\bar{S}, \bar{t}, n)$ denotes the price of a down-and-out barrier option on a non dividend-paying equity at time $\bar{t}$ and asset price $\bar{S}$ with the monitoring dates $0 = \bar{t}_0 < \bar{t}_1 < ... < \bar{t}_N = \bar{T}$ that take at necessarily equally-spaced points in time and option maturity $\bar{T}$ and the constant lower barrier $\bar{L}$ and the strick price $\bar{K}$. The asset price $\bar{S}$ satisfies the following geometric Brownian motion process.

$$\frac{d\bar{S}}{\bar{S}}=r_c dt+\sigma_c dB$$

with the constant volatility σ_c and the constant risk-free rate r_c, then $\bar{V}(\bar{S}, \bar{t}, n)$ satisfies Black-Scholes partial differential equation:

$$-\frac{\partial \bar{V}(\bar{S}, \bar{t}, n)}{\partial \bar{t}}+r_c\bar{S}\frac{\partial \bar{V}(\bar{S}, \bar{t}, n)}{\partial \bar{S}}+\frac{\sigma_c^2\bar{S}^2}{2}\frac{\partial^2 \bar{V}(\bar{S}, \bar{t}, n)}{\partial \bar{S}^2}=r_c\bar{V}(\bar{S}, \bar{t}, n) \tag{4}$$

As before if the trigger conditions update at the monitoring dates, then, we have:

$$\bar{V}(\bar{S}, \bar{t}_n, n) = \bar{V}(\bar{S}, \bar{t}_n, n-1)I_{(\bar{S}\geq \bar{L})} \tag{5}$$

$$\bar{V}(\bar{S}, \bar{t}_0, 0) = (\bar{S}-\bar{K})I_{(\bar{S}\geq \max\{\bar{K},\bar{L}\})} \tag{6}$$

Fusai et al., (2006) provided an analytical approach to evaluate $\bar{V}(\bar{S}, \bar{t}_n, n)$ under the partial differential Equation (4) with boundary conditions (5) and (6). The following provides definition of some auxiliary functions that are used in the formulation of Fusai et al. (2006). For the asset price $\bar{S}$ the function $C_{BS}(\bar{S})$ is defined as follows:

$$C_{BS}(\bar{S})=\left[\bar{S}\,N(d_1)-\bar{K}e^{-r_c\bar{t}_n}N(d_2)\right]I_{[0,\infty)}(k)+\left[\bar{S}-\bar{K}e^{-r_c\bar{t}_n}\right]I_{(-\infty,0]}(k),$$

in which

$$d_1 = \frac{\ln(\frac{\bar{s}}{\bar{k}}) + (r_c + \frac{\sigma_c^2}{2})(\bar{t}_n - \bar{t}_0)}{\sigma_c \sqrt{\bar{t}_n - \bar{t}_0}}; d_2 - d_1 = \sigma_c \sqrt{\bar{t}_n - \bar{t}_0}; k = \ln(\frac{\bar{k}}{\bar{L}}),$$

And the complex coefficients μ_n is defined as below:

$$\mu_n = \sqrt{\ln q + 2n\pi i}, \qquad n \in Z,$$

in which q must satisfy the following condition:

$$|q| \prec \exp\left\{-(1-\alpha)^2 \gamma^2 I_{((1-\alpha)\gamma \geq 0)}\right\},$$

where

$$\alpha = -\frac{r_c - \frac{\sigma_c^2}{2}}{\sigma_c^2}; \tau = \bar{t}_{n+1} - \bar{t}_0; \gamma = \frac{\sigma_c \sqrt{\tau}}{\sqrt{2}}.$$

Also the functions L+(u) and ˘ F (z, q) are respectively defined as follows:

$$L_+(u) = \exp(\frac{u}{\pi i} \int_0^\infty \frac{\ln\left(1 - q e^{-z^2}\right)}{z^2 - u^2} dz), \Im(u) \succ 0,$$

$$\breve{F}(z,q) = \frac{-i\bar{L}\gamma}{4} e^{(1-\alpha)k} \sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n) e^{i\mu_n \frac{z}{\gamma}}}{\mu_n} \sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n) e^{i\mu_n \frac{z}{\gamma}}}{\mu_m(\mu_m + i\alpha\gamma)(\mu_m + i(\alpha-1)\gamma(\mu_m + \mu_n))} I_{(k \geq 0)}$$

$$-\frac{\bar{L}}{2} \sum_{n=-\infty}^{\infty} \frac{L_+(\mu_n) e^{i\mu_n \frac{z}{\gamma}}}{\mu_n} \left(\frac{e^k}{L_+(i\alpha\gamma)(\mu_n - i\alpha\gamma)} - \frac{1}{L_+(i(\alpha-1)\gamma)(\mu_n - i(\alpha-1)\gamma)} I_{(k \prec 0)}\right.$$

The general price of down-and-out discrete barrier option under the partial differential Equation (4) with boundary conditions (5) and (6) is stated in the following theorem.

Theorem 1. (Fusai et al., 2006): Consider partial differential Equation (4) with boundary conditions (5) and (6). Then,

(i) The price of down-and-out discrete barrier option at a monitoring date tn and asset price S is given by

$$\bar{V}(\bar{S}, \bar{t}_n, n) = C_{BS}(\bar{S})(\frac{\bar{S}}{\bar{L}})^{\alpha} e^{\beta \bar{t}_n} \breve{f}(\ln\frac{\bar{S}}{\bar{L}}, n), \tag{7}$$

where $\breve{f}$ and β is defined as follows

$$\breve{f}(z,n) = \frac{1}{2\pi\rho^n} \int_0^{2\pi} \breve{F}(z, \rho e^{iu}) e^{inu} du,$$

$$\beta = \alpha m + \alpha^2 \frac{\sigma_c^2}{2} - r_c$$

(ii) The Greeks of the contract, namely Delta, $\frac{\partial \bar{V}(\bar{S}, \bar{t}_n, n)}{\partial \bar{S}}$, and Gamma, $\frac{\partial^2 \bar{V}(\bar{S}, \bar{t}_n, n)}{\partial \bar{S}^2}$ are obtained as below

$$\Delta = \Delta_{BS}(\bar{S}) + \frac{1}{L}(\frac{\bar{S}}{\bar{L}})^{\alpha-1} e^{\beta \bar{t}_n} \left[\alpha \breve{f}(z,n) + \frac{\partial \breve{f}(z,n)}{\partial z}\right], \tag{8}$$

With $z=\ln(\frac{\overline{S}}{\overline{L}})$ and where $\Delta_{BS}(\overline{S}) = N(d_1)I_{[0,\infty)}(k) + I_{(-\infty,0]}(k)$

$$\Gamma = \Gamma_{BS}(\overline{S}) + \frac{1}{\overline{L}^2}(\frac{\overline{S}}{\overline{L}})^{\alpha-2} e^{\beta t_n}\left[\alpha(\alpha-1)\breve{f}(z,n) + (2\alpha-1)\frac{\partial \breve{f}(z,n)}{\partial z} + \frac{\partial^2 \breve{f}(z,n)}{\partial z^2}\right], \quad (9)$$

where $\Gamma_{BS}(\overline{S})$ is the Black-Scholes Gamma.

The corresponding down-and-in call option can be priced by subtracting from the price of standard call the price of the down-and-out call. Likewise, the barrier put option can be priced using the put call transformation given in Haug (1999).

Now consider, the problem of pricing the down-and-out discrete barrier option under the partial differential Equation (1), i.e., the problem of evaluating $V(S, t_n, n)$ under the partial differential Equation (1) with conditions (2) and (3). The next section provides an analytic solution for such problem.

3. Results

This section provides an exact and analytic solution for the problem of pricing the down-and-out discrete barrier option under time-dependent framework. The following theorem provides the main result of this article.

Theorem 2. Suppose $V(S, t, n)$ represents the price of down-and-out barrier option at the monitoring date t_n and the asset price S which satisfies the partial differential Equation (1) and conditions (2) and (3). Then,

$$V(S, t_n, n) = \frac{K}{\overline{K}}\exp\left(\int_0^{t_n}\left[r_c\frac{\sigma^2}{\sigma_c^2} - r(u)\right]du\right)\times \overline{V}(\overline{S}, \overline{t}_n, n), \quad (10)$$

where $\overline{V}(\overline{S}, \overline{t}_n, n)$ is the price of discrete barrier option that is calculated based upon Theorem 1. The above $\overline{V}(\overline{S}, \overline{t}_n, n)$ with strick price $\overline{K}$ and lower barrier $\overline{L}$ and option maturity $\overline{T}$ that satisfy three following conditions.

$$\max\left\{\frac{\overline{L}}{\overline{K}}, 1\right\} = \max\left\{\frac{L}{K}, 1\right\} \quad (11)$$

$$\frac{\overline{L}}{\overline{K}} = \frac{L}{K}\exp\left(\int_0^{t_n}\left[r(u) - d(u) - r_c\frac{\sigma^2}{\sigma_c^2}\right]du\right) \quad (12)$$

$$\overline{T} = T\frac{\sigma^2}{\sigma_c^2} \quad (13)$$

Also, $\overline{t}_n$ and $\overline{S}$ in the Equation (10) are calculated as follows:

$$\overline{t}_n = t_n\frac{\sigma^2}{\sigma_c^2} \quad (14)$$

$$\overline{S} = \frac{\overline{K}}{K}\exp\left(\int_0^{t_n}\left[r(u) - d(u) - r_c\frac{\sigma^2}{\sigma_c^2}\right]du\right)\times S \quad (15)$$

Proof. In the first step, we transform the Black-Scholes equation with the time-varying parameters (1) into Black-Scholes equation (4) such that:

$$\overline{t} = 0 \text{ when } t = 0 \quad (16)$$

To observe this, we use the following transformations:

$$\mathrm{V}(\mathrm{S},\mathrm{t},\mathrm{n})=\mathrm{h}(\mathrm{t})\overline{\mathrm{V}}(\overline{\mathrm{S}},\bar{\mathrm{t}}_{n},\mathrm{n}) \quad , \quad \overline{\mathrm{S}}=\phi(t)\mathrm{S} \quad , \quad \bar{\mathrm{t}}=\psi(t)\mathrm{t} \tag{17}$$

Using the chain rule, one may conclude that:

$$\frac{\partial \mathrm{V}}{\partial \mathrm{t}}=\mathrm{h}(\mathrm{t})\left[\frac{\partial \overline{\mathrm{V}}}{\partial \overline{\mathrm{S}}}\phi'(t)S+\frac{\partial \overline{\mathrm{V}}}{\partial \bar{\mathrm{t}}}\psi'(t)\right]+\mathrm{h}'(\mathrm{t})\overline{\mathrm{V}}$$

$$\frac{\partial \mathrm{V}}{\partial \mathrm{S}}=\mathrm{h}(\mathrm{t})\phi(t)\frac{\partial \overline{\mathrm{V}}}{\partial \overline{\mathrm{S}}},$$

$$\frac{\partial^2 \mathrm{V}}{\partial \mathrm{S}^2}=\mathrm{h}(\mathrm{t})\phi(t)^2\frac{\partial^2 \overline{\mathrm{V}}}{\partial \overline{\mathrm{S}}^2},$$

Substituting the above equations into (1) leads to:

$$-\frac{\partial \overline{\mathrm{V}}}{\partial \bar{\mathrm{t}}}+\left(\frac{[r(t)-d(t)]\phi(t)S-\phi'(t)S}{\psi'(t)}\right)\frac{\partial \overline{\mathrm{V}}}{\partial \overline{\mathrm{S}}}+\left(\frac{\sigma^2 S^2\phi(t)^2}{2\psi'(t)}\right)\frac{\partial^2 \overline{\mathrm{V}}}{\partial \overline{\mathrm{S}}^2}-\left(\frac{\mathrm{h}'(\mathrm{t})+\mathrm{r}(\mathrm{t})\mathrm{h}(\mathrm{t})}{\mathrm{h}(\mathrm{t})\phi'(t)}\right)\overline{\mathrm{V}}=0$$

Comparing this with Equation (4) yields:

$$\frac{\sigma_c^{\,2}}{2}\overline{S}^2=\frac{\sigma_c^{\,2}S^2\phi(t)^2}{2\psi'(t)} \tag{18}$$

$$r_c\overline{S}=\frac{[r(t)-d(t)]\phi(t)S-\phi'(t)S}{\psi'(t)} \tag{19}$$

$$r_c=\frac{h'(t)+r(t)h(t)}{h(t)\psi'(t)} \tag{20}$$

From Equation (17) to (20), one may conclude that:

$$\psi(t)=\frac{1}{\sigma_c^{\,2}}\int_0^t \sigma^2 d(u)+A \tag{21}$$

$$\phi(t)=\beta\exp\int_0^t [r(u)-d(u)-r_c\psi'(u)]du, \tag{22}$$

$$h(t)=C\exp\int_0^t [r_c\psi'(u)-r(u)]du, \tag{23}$$

where A , B , and C are constant.

Up to now, the Black-Scholes equation with time-varying parameters (1) is transformed into Black-Scholes partial differential equation (4). As mentioned before, $\overline{\mathrm{V}}(\overline{\mathrm{S}},\bar{\mathrm{t}}_n,\mathrm{n})$ with the two conditions (5), (6) can be evaluated based on Theorem 1. Therefore, the constants A , B , and C are determined and also some conditions on L , K are given simultaneously such that conditions (5), (6) equivalent to the (2) and (3).

Finally, $\mathrm{V}(\mathrm{S},\mathrm{t},\mathrm{n})$ under the partial differential equations (1) with conditions (2) and (3) is evaluated with the first equation in (17).

From (3), (17) and (6) we have:

$$\begin{aligned}(\mathrm{S}-\mathrm{K})^{+}\mathrm{I}_{[\max\{\mathrm{K},\mathrm{L}\},\infty)}(\mathrm{S}) &= \mathrm{V}(\mathrm{S},\mathrm{t}_0,0)\\ &= \mathrm{h}(\mathrm{t}_0)\overline{\mathrm{V}}(\overline{\mathrm{S}},\bar{\mathrm{t}}_0,0)\\ &= \mathrm{h}(\mathrm{t}_0)(\overline{\mathrm{S}}-\overline{K})^{+}\mathrm{I}_{[\max\{\overline{\mathrm{K}},\overline{\mathrm{L}}\},\infty)}(\overline{\mathrm{S}})\end{aligned}$$

But

$$h(t_0)(\bar{S} - \bar{K})^{+}I_{[\max\{K,L\},\infty)}(\bar{S}) = h(t_0)(\phi(t_0)S\text{-}\bar{K})^{+}I_{[\max\{\bar{K},\bar{L}\},\infty)}(\phi(t_0)S)$$

$$= h(t_0)\phi(t_0)\,(S\text{-}\frac{\bar{K}}{\phi(t_0)})^{+}I_{[\frac{\max\{\bar{K},\bar{L}\}}{\phi(t_0)},\infty)}(S)$$

Therefore

$$h(t_0)\phi(t_0)=1 \quad , \quad K=\frac{\bar{K}}{\phi(t_0)} \quad , \quad I_{[\max\{K,L\},\infty)}(S)=I_{[\frac{\max\{\bar{K},\bar{L}\}}{\phi(t_0)},\infty)}(S) \tag{24}$$

Using Equations (16) to (24), one may conclude that:

$$A=\psi(t_0)=0 \tag{25}$$

$$B=\frac{\bar{K}}{K}=\phi(t_0) \tag{26}$$

$$C=\frac{K}{\bar{\bar{K}}}=h(t_0) \tag{27}$$

$$\frac{\max\{\bar{K},\bar{L}\}}{\bar{K}}=\frac{\max\{K,L\}}{K} \tag{28}$$

Multiplying the both side of equation (5) by $h(t_n)$ leads to

$$V(S,t_n,n)=V(S,t_n,n-1)I_{[\bar{L},\infty)}(\bar{S}) \tag{29}$$

Equations (29) and (2) are equivalent, whenever the following conditions on $\bar{L}$ and $\bar{K}$ are exist.

$$I_{[\bar{L},\infty)}(\bar{S})=I_{[L,\infty)}(S)$$

Therefore, $\bar{L}$ and $\bar{K}$ are chosen so that $(\bar{S}\ \geq\ \bar{L})$ equivalent $(S\ \geq\ L)$. Then, with equations (17), (22) and (25) it is sufficient to have

$$\frac{\bar{L}}{\bar{\bar{K}}}=\frac{L}{K}\exp\int_0^{t_n}\left[r(u)-d(u)-r_c\frac{\sigma^2}{\sigma_c^2}\right]du \tag{30}$$

Also we must aware of accuracy of the $(\bar{S}\ \geq\ \bar{L})$, then $\bar{L}$ and $\bar{K}$ is chosen as following

$$\frac{\bar{L}}{\bar{\bar{K}}}\leq\frac{S}{K}\exp\int_0^{t_n}\left[r(u)-d(u)-r_c\frac{\sigma^2}{\sigma_c^2}\right]du \tag{31}$$

This observation complete the proof.

It would be worthwhile to mention: (i) two constant parameters r_c and σ^2, given in Theorem 2, respectively, are arbitrary risk-free and volatility in the Black-Scholes partial differential Equation (4); (ii) There is a fixed period between the monitoring dates $\bar{t}_n$ that is driven in the theorem 2. If the volatility is non-constant function of time then the period between monitoring dates t_n may be not fixed and the Theorem 1 is unusable in this place.

The following evaluates the Greeks of contract.

Corollary 1. The Greeks of the contract, namely Delta $\frac{\partial V(S,t_n,n)}{\partial S}$ and Gamma, $\frac{\partial^2 V(S,t_n,n)}{\partial S^2}$, under the Black-Scholes equation with time-varying parameters (1) with conditions (2), (3) are obtained as follows:

$$\Delta=\exp\int_0^{t_n}\left[r_c\frac{\sigma^2}{\sigma_c^2}-r(u)\right]du\times\exp\int_0^{t_n}\left[r(u)-d(u)-r_c\frac{\sigma^2}{\sigma_c^2}\right]du\times\frac{\partial\bar{V}(\bar{S},\bar{t}_n,n)}{\partial\bar{S}} \tag{32}$$

$$\Gamma = \frac{\overline{K}}{K}\exp\int_0^{t_n}\left[r_c\frac{\sigma^2}{\sigma_c^2}-r(u)\right]du \times (\exp\int_0^{t_n}\left[r(u)-d(u)-r_c\frac{\sigma^2}{\sigma_c^2}\right]du)^2 \times \frac{\partial^2 \overline{V}(\overline{S},\overline{t}_n,n)}{\partial \overline{S}^2} \tag{33}$$

where $\frac{\partial \overline{V}(\overline{S},\overline{t}_n,n)}{\partial \overline{S}}$ and $\frac{\partial \overline{V}(\overline{S},\overline{t}_n,n)}{\partial \overline{S}}$ are evaluated according to the Theorem 1(ii).

***Proof*.** Desire proof arrives by derivation with respect to S from Equation (10) and (15).

The following proves that the price of discrete barrier option and the Greeks of the contract are invariant from choosing $\overline{L}$ and $\overline{K}$ in Theorem 2.

Corollary 2. For the given lower barrier L and the stirk price K, there may exist many of the $\overline{L}$ and $\overline{K}$ such that satisfy conditions (11), (12). But the price of discrete barrier option and the Greeks of the contract that respectively evaluate based on (10), (32) and (33) are independent of different choices for $\overline{L}$ and $\overline{K}$.

Proof. We just prove the corollary for the price of barrier option, the result for the Greeks consequentis immediate. Without less of generality suppose that $L \prec K$. By replacing the formulae with $\overline{V}(\overline{S},\overline{t}_n,n)$ given by the Theorem 1 into Theorem 2, one may observe.

$$\overline{V}(\overline{S},\overline{t}_n,n) \propto \frac{K}{\overline{K}}\left[C_{BS}(\overline{S}) + (\frac{\overline{S}}{\overline{L}})^{\alpha} e^{\beta \overline{t}_n} \breve{f}(\ln\frac{\overline{S}}{\overline{L}},n)\right]$$

$$\propto K\frac{\overline{S}}{\overline{K}}N(d_1) - Ke^{-r_c\bar{t}_n}N(d_2) + \frac{K}{\overline{K}}(\frac{\overline{S}}{\overline{L}})^{\alpha} e^{\beta \overline{t}_n}\breve{f}(\ln\frac{\overline{S}}{\overline{L}},n)$$

Now using Equation (15), the last expression can be restated as:

$$K\phi^*(t_n)SN(d_1) - Ke^{-r_c\bar{t}_n}N(d_2) + (\frac{1}{\overline{K}})^{\alpha-1}(\phi^*(t_n)S)^{\alpha}\frac{1}{\overline{K}}(\frac{\overline{K}}{\overline{L}})^{\alpha} e^{\beta \overline{t}_n}\breve{f}(\ln\frac{\overline{K}}{\overline{L}} + \ln\frac{\phi^*(t_n)}{K}S,n) \tag{34}$$

where

$$\phi^*(t_n) = \exp\int_0^{t_n}\left[r(u)-d(u)-r_c\frac{\sigma^2}{\sigma_c^2}\right]du$$

The function $\breve{f}(z,q)$ that was defined previously has lower barrier $\overline{L}$ as coefficient. Then, the last equation depend on $\overline{L}$ and $\overline{K}$ only from the proportion $\frac{\overline{L}}{\overline{K}}$. This fact completes the desire proof.

The following examples provide application of the above results to the problem of pricing a discrete barrier option and the Greeks of the contract for the different cases of $r(t)$ and $d(t)$.

Example 1. Suppose, we want to price down-and-out barrier option. Parameters used are $S=100$, $K=100$, $r(t) = 0.1+0.05\exp(-t)$, $d(t) \equiv 0.05$, $\sigma=0.2$, $T = 0.5$. Also the arbitrary parameters in the transformed partial differential equation are $r_c = 0.1$, $\sigma_c=0.3$. The price of the barrier option and the Greeks of the contract at the last monitoring date are evaluated for the different lower barriers L and the different monitoring numbers N. Results are summarized in Table 1. In this example for lower barrier greater than (97) there are no $\overline{L}$ and $\overline{K}$ such that satisfy conditions (11) and (12).

Table 1: In this table for the different levels of the lower barrier, we chose $\overline{L}$ and $\overline{K}$ suitably, and then the price of discrete barrier options is evaluated based on Theorem 2

N	L	V	Δ	Γ
5	88	7.6920	0.6515	0.0222
5	91	7.4905	0.6836	0.0188
5	94	7.0023	0.7375	0.0189
5	97	6.0966	0.7859	0.0403
15	88	7.6307	0.6622	0.0204
15	91	7.3110	0.7124	0.0132
15	94	6.5459	0.8121	0.0041
15	97	5.0815	0.9251	0.0266

Example 2. Consider the problem of pricing under $S=100$, $K=100$, $r(t) = 0.075+0.05\,t$, $d(t) \equiv 0.03+0.02t$, $\sigma=0.3$, $T = 0.2$. Also the arbitrary parameters r_c and σ_c are given as previous example. The price of the barrier option and the Greeks of the contract at the last monitoring date are evaluated for the different lower barriers L and the different monitoring numbers N. Results are summarized in Table 2.

Table 2: In this table for the different levels of the lower barrier, we chose $\overline{L}$ and $\overline{K}$ suitably and then the price of discrete barrier options are evaluated based on theorem 2

N	L	V	Δ	Γ
5	87	5.7588	0.5562	0.0281
5	91	5.6456	0.5780	0.0249
5	95	5.1946	0.6315	0.0253
5	99	4.1134	0.8681	-0.7255
10	87	5.7489	0.5583	0.0277
10	91	5.5871	0.5896	0.0221
10	95	4.9358	0.6804	0.0175
10	99	3.4266	0.7254	-0.4748

4. Discussions

This article studies the pricing of discrete barrier options under a model where the risk-free rate, $r(t)$, and the dividend yield, $d(t)$, are two given the functions of time. For non-constant volatility $\sigma(t)$, one may use the average volatility. Then, apply findings of this article to such problem. Also, if the problem of option pricing can be solved when the monitoring dates take at not necessarily equally-spaced point, then the approach of the present paper is usable for the non-constant volatility too.

Acknowledgements:

The author would like to thank professor Fussai and miss Hodoudi for their useful comments.

Corresponding Author:
Mohammad H. Beheshti
Department of Statistics, Science and Research Branch, Islamic Azad University, Tehran, Iran.
mohammadhoseinbeheshti@yahoo.com

References

1 Ait-Sahlia, F. & Lai, T. L. (1997). Valuation of discrete barrier and hindsight options. J. Financial Eng. 6, 169–77.
2 Bertoldi, M. & Bianchetti, M. (2003). Monte Carlo simulation of discrete barrier options. Internal Report, Financial Engineering-Derivatives Modelling, Caboto SIM S.p.a., Banca Intesa Group, Milan, Italy.
3 Broadie, M., Glasserman, P. & Kou. S. (1997). A continuity correction for discrete barrier options. Math Finance. 7, 325–349.
4 Fusai, G., Abrahams, I. D. & Sgarra, C. (2006). An exact analytical solution for barrier options, Financ Stochast. 10, 1–26.
5 Geman. H. & Yor, M. (1996). Pricing & hedging double barrier options: a probabilistic approach. Math Finance.6, 365–378.
6 Haug, E. G. (1999). Barrier put-call transformations, Tempus Financial Engineering

Number 3–97, Norway, download at http://ssrn.com/abstract=150156.
7 Heynen R. C. & Kat, H. M. (1995). Lookback options with discrete and partial monitoring of the underlying price. Appl. Math. Finance.2, 273–284.
8 Lo, C. F., Lee, H. C. & Hui, C. H. (2003). A simple approach for pricing barrier options with time-dependent parameters. Quantitative finance. 3, 98–107.
9 Marianito, R.R. & Rogemar, S. M (2006). An alternative approach to solving the Black-Sholes equation with time-varing parameters. Applied Mathematics Letters. 19, 398–402.
10 Merton, R. (1973). Theory of rational option pricing, Bell Journal of Management Sciences.4, 141–183.
11 Pelsser, A. (2000). Pricing double barrier options using Laplace transforms. Finance Stochast. 4, 95–104.
12 Rich, D. R. (1994). The mathematical foundations of barrier option pricing theory. Adv. Futures Options Res. 7, 267–311.
13 Roberts, G. O. & Hortland C. F. (1994). The hazard rate tangent approximation for boundary hitting times. Ann. Appl. Prob. 5, 446–60.
14 Wilmott, P. Howison, S. & Dewynne, J. (1999). The Mathematics of Financial Derivatives, Cambridge, University Press.

8/25/2020