**An Analytical Approach to Pricing Discrete Barrier Options under Time-dependent Models**

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**Abstract:** Consider the problem of pricing a discrete barrier option under a time-dependent framework. This article provides an analytical solution for such an interesting problem in two steps. Namely, in the first step, the problem in hand restates a time invariant which has an exact solution. Secondly, the exact solution for the time-dependent model arrives by substituting such a solution in an integral equation. Applications to the Greeks of the contracts are given.

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**Keywords:** Barrier option, Black-Scholes framework, Discrete monitoring, Time-dependent model.

**1. Introduction**

The problem of pricing a discrete barrier option plays a role in the quantitative finance and financial industry. Barrier options are usually traded as the modification of simple European puts and call options. Barrier options activated (knock-ins) or terminated (knock-outs) if the sample path of the underlying asset has crossed a predetermined barrier prior to the exercise time. There are several pricing formulaes for barrier options in the Black-Scholes framework (see, Plesser 2000; Geman & Yor 1996; Rich 1994). In practice, barrier options differ from those studied in the academic literature in many respects. One of the most important is the monitoring frequency of the underlying assets. In the case of discrete monitoring, the sample path of the underlying asset is monitored at fixed times. Heynen & Kat (1995) were probably the scholars who issued an article noticing the discrepancy between option price under continuous and discrete monitoring. After seminal work of Heynen & Kat (1995), several authors proposed approximations based on a variety of different numerical approaches (see, Ait-sahlia & Lai, 1997; Bertoldi & Bianchetti, 2003; and Broadie & Glasserman, 1997). They used several methods such as: Recursive integration method, monte carlosimulation and trinomial tree. In the numerical approaches, computational cost increases whenever the number of monitoring increases. Moreover, the accuracy of the approximated solution decreases whenever asset price and the barrier are close to each other, in some sense. To overcome such difficulties, Fusai et al. (2006) provided an analytical method for the problem of pricing a discrete barrier option under time invariant framework.

For the problem of pricing a discrete barrier option under time-dependent framework (time-dependent parameters) the above findings are not valid anymore. More precisely, the problem of pricing a barrier option with time-dependent parameters is not a trivial extension of time invariant model. Roberts & Shorthland (1997) applied the hazard rate tangent approximation method to evaluate upper and lower bounds for price of such time-dependent barrier option. Unfortunately, their bounds cannot state in the closed form and consequently cannot be improved. Lo et al. (2003) presented a simple approach for computing upper and lower bounds (in the close form) for the price of a barrier option.

This article considers the problem of pricing a down-and-out discrete barrier options on a divided paying equity whenever the risk-free rate and the dividend yield in the Black-Scholes partial differ- essential equation are deterministic functions of time. A method of reducing time-dependent partial differential equation to the heat equation is described in Wilmott et al. (1999). Also Marianito & Rogemar (2006) outlined a procedure that transforms the Black-Scholes partial differential equation with time-dependant parameters into the Black-Scholes equation with time-independent parameters. Our approach is to reduce the pricing problem to the time-independent case and the solution to the latter equation provided by Fusai et al. (2006).

Section 2 collects some useful elements for the other sections. Section 3 provides the price of discrete barrier option under the time-dependent parameters. The price of the Greeks of contract along with two examples are given in Section 3.

**2. Material and Methods**

Now, we collect some essential elements for the next sections. Suppose  be the monitoring dates that take at necessarily equally-spaced points in time,  the option maturity, the L the constant lower barrier which is active at all times tn and  is the strick price of the option. The nth time interval is defined as . We are interested in pricing a down-and-out call option, i.e., a call option that expires worthless if a lower barrier has been hit at monitoring date. We denote is the price of a down and-out barrier option on a dividend-paying equity at time t in the nth time interval and the asset price . The asset price  satisfies the following stochastic differential equation.



with the constant volatility , the risk-free rate and the dividend yield , then it is well known (see, Merton (1973)) that  satisfies following partial differential equation:

 (1)

Given that the trigger condition is checked only at fixed times, it is needed to update the initial condition at each of the monitoring dates :

(2)

 (3)

Now, suppose  denotes the price of a down-and-out barrier option on a non dividend- paying equity at timeand asset price  with the monitoring dates that take at necessarily equally-spaced points in time and option maturity  and the constant lower barrier  and the strick price . The asset price  satisfies the following geometric Brownian motion process.



with the constant volatility  and the constant risk-free rate , then satisfies Black-Scholes partial differential equation:

 (4)

As before if the trigger conditions update at the monitoring dates, then, we have:

 (5)

 (6)

Fusai et al., (2006) provided an analytical approach to evaluate  under the partial differential Equation (4) with boundary conditions (5) and (6). The following provides definition of some auxiliary functions that are used in the formulation of Fusai et al. (2006). For the asset price  the function is defined as follows:

,

in which



And the complex coefficients  is defined as below:



in which  must satisfy the following condition:



where

.

Also the functions L+(u) and ˘ F (z, q) are respectively defined as follows:





The general price of down-and-out discrete barrier option under the partial differential Equation (4) with boundary conditions ( 5) and (6) is stated in the following theorem.

**Theorem 1.** (Fusai et al., 2006): Consider partial differential Equation (4) with boundary conditions (5) and (6). Then,

**(i)** The price of down-and-out discrete barrier option at a monitoring date tn and asset price S is given by

 (7)

where and  is defined as follows





**(ii)** The Greeks of the contract, namely Delta, , and Gamma, 

are obtained as below

 (8)

With and where 

 (9)

where  is the Black-Scholes Gamma.

The corresponding down-and-in call option can be priced by subtracting from the price of standard call the price of the down-and-out call. Likewise, the barrier put option can be priced using the put call transformation given in Haug (1999).

Now consider, the problem of pricing the down-and-out discrete barrier option under the partial differential Equation (1), i.e., the problem of evaluating under the partial differential Equation (1) with conditions (2) and (3). The next section provides an analytic solution for such problem.

**3. Results**

This section provides an exact and analytic solution for the problem of pricing the down-and-out discrete barrier option under time-dependent framework. The following theorem provides the main result of this article.

**Theorem 2.** Suppose  represents the price of down-and-out barrier option at the monitoring date and the asset price  which satisfies the partial differential Equation (1) and conditions (2) and (3). Then,

 (10)

where is the price of discrete barrier option that is calculated based upon Theorem 1. The above with strick price and lower barrier  and option maturity that satisfy three following conditions.

 (11)

 (12)

 (13)

Also, and in the Equation (10) are calculated as follows:

 (14)

 (15)

***Proof.*** In the first step, we transform the Black-Scholes equation with the time-varying parameters (1) into Black-Scholes equation (4) such that:

when  (16)

To observe this, we use the following transformations:

 (17)

Using the chain rule, one may conclude that:







Substituting the above equations into (1) leads to:



Comparing this with Equation (4) yields:

 (18)

 (19)

 (20)

From Equation (17) to (20), one may conclude that:

 (21)

 (22)

 (23)

where , , and  are constant.

Up to now, the Black-Scholes equation with time-varying parameters (1) is transformed into Black-Scholes partial differential equation (4). As mentioned before, with the two conditions (5), (6) can be evaluated based on Theorem 1. Therefore, the constants , , and are determined and also some conditions on , are given simultaneously such that conditions (5), (6) equivalent to the (2) and (3).

Finally, under the partial differential equations (1) with conditions (2) and (3) is

evaluated with the first equation in (17).

From (3), (17) and (6) we have:



But



Therefore

 (24)

Using Equations (16) to (24), one may conclude that:

 (25)

 (26)

 (27)

 (28)

Multiplying the both side of equation (5) by leads to

 (29)

Equations (29) and (2) are equivalent, whenever the following conditions on and are exist.



Therefore, and are chosen so that  equivalent . Then, with equations (17), (22) and (25) it is sufficient to have

 (30)

Also we must aware of accuracy of the , then  and is chosen as following

 (31)

This observation complete the proof.

It would be worthwhile to mention: (i) two constant parameters and , given in Theorem 2, respectively, are arbitrary risk-free and volatility in the Black-Scholes partial differential Equation (4); (ii) There is a fixed period between the monitoring dates  that is driven in the theorem 2. If the volatility is non-constant function of time then the period between monitoring dates may be not fixed and the Theorem 1 is unusable in this place.

The following evaluates the Greeks of contract.

**Corollary 1.** The Greeks of the contract, namely Delta and Gamma, , under the Black-Scholes equation with time-varying parameters (1) with conditions (2), (3) are obtained as follows:

 (32)

 (33)

where and  are evaluated according to the Theorem 1(ii).

***Proof***. Desire proof arrives by derivation with respect to  from Equation (10) and (15).

The following proves that the price of discrete barrier option and the Greeks of the contract are invariant from choosing and in Theorem 2.

**Corollary 2.** For the given lower barrier and the stirk price , there may exist many of the and such that satisfy conditions (11), (12). But the price of discrete barrier option and the Greeks of the contract that respectively evaluate based on (10), (32) and (33) are independent of different choices for and .

***Proof.*** We just prove the corollary for the price of barrier option, the result for the Greeks consequentis immediate. Without less of generality suppose that . By replacing the formulae with  given by the Theorem 1 into Theorem 2, one may observe.



Now using Equation (15), the last expression can be restated as:

 (34)

where



The function  that was defined previously has lower barrier  as coefficient. Then, the last equation depend on and only from the proportion . This fact completes the desire proof.

The following examples provide application of the above results to the problem of pricing a discrete barrier option and the Greeks of the contract for the different cases of  and .

**Example 1.** Suppose, we want to price down-and-out barrier option. Parameters used are , , , , , . Also the arbitrary parameters in the transformed partial differential equation are , . The price of the barrier option and the Greeks of the contract at the last monitoring date are evaluated for the different lower barriers  and the different monitoring numbers . Results are summarized in Table 1. In this example for lower barrier greater than (97) there are no  and  such that satisfy conditions (11) and (12).

**Table 1**: In this table for the different levels of the lower barrier, we chose  and suitably, and then the price of discrete barrier options is evaluated based on Theorem 2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| **5** | 88 | 7.6920 | 0.6515 | 0.0222 |
| **5** | 91 | 7.4905 | 0.6836 | 0.0188 |
| **5** | 94 | 7.0023 | 0.7375 | 0.0189 |
| **5** | 97 | 6.0966 | 0.7859 | 0.0403 |
| **15** | 88 | 7.6307 | 0.6622 | 0.0204 |
| **15** | 91 | 7.3110 | 0.7124 | 0.0132 |
| **15** | 94 | 6.5459 | 0.8121 | 0.0041 |
| **15** | 97 | 5.0815 | 0.9251 | 0.0266 |

**Example 2**. Consider the problem of pricing under , , , , , . Also the arbitrary parameters and are given as previous example. The price of the barrier option and the Greeks of the contract at the last monitoring date are evaluated for the different lower barriers and the different monitoring numbers . Results are summarized in Table 2.

**Table 2:** In this table for the different levels of the lower barrier, we chose  and suitably and then the price of discrete barrier options are evaluated based on theorem 2

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
|  |  |  |  |  |
| **5** | 87 | 5.7588 | 0.5562 | 0.0281 |
| **5** | 91 | 5.6456 | 0.5780 | 0.0249 |
| **5** | 95 | 5.1946 | 0.6315 | 0.0253 |
| **5** | 99 | 4.1134 | 0.8681 | -0.7255 |
| **10** | 87 | 5.7489 | 0.5583 | 0.0277 |
| **10** | 91 | 5.5871 | 0.5896 | 0.0221 |
| **10** | 95 | 4.9358 | 0.6804 | 0.0175 |
| **10** | 99 | 3.4266 | 0.7254 | -0.4748 |

**4. Discussions**

This article studies the pricing of discrete barrier options under a model where the risk-free rate, , and the dividend yield, , are two given the functions of time. For non-constant volatility , one may use the average volatility. Then, apply findings of this article to such problem. Also, if the problem of option pricing can be solved when the monitoring dates take at not necessarily equally-spaced point, then the approach of the present paper is usable for the non-constant volatility too.

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