A Laplace Transformation approach to Simultaneous Linear Differential Equations

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Abstract: The Laplace transformation is a mathematical tool which is used in the solving of differential equations by converting it from one form into another form. Regularly it is effective in solving linear differential equations either ordinary or partial. The Laplace transformation is used in solving the time domain function by converting it into frequency domain function. Laplace transformation makes it easier to solve the problem in engineering application and make differential equations simple to solve. In this paper we will discuss How to approach the Laplace Transformation to Simultaneous Linear Differential Equations.
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Sub area: Laplace transformation
Broad area: Mathematics

Introduction:

The Laplace transformation is applied in different areas of science, engineering and technology. The Laplace transformation is applicable in so many fields. The Laplace Transform was primary used and named after by Pierre Simon Laplace Pierre Simon Laplace was a French Mathematician an Astronomer, who had a lot of control in the growth of several theories in mathematics, statistics, physics, and astronomy. He contributed seriously to physical mechanics, by converting the previous geometrical analysis to one based on calculus, which opened up application of his formulas to a wider range of problems. It is effective in solving linear differential equation either ordinary or partial. It reduces an ordinary differential equation into algebraic equation. Ordinary linear differential equation with constant coefficient and variable coefficient can be easily solved by the Laplace transformation method without finding the generally solution and the arbitrary constant. It is used in solving Physical problems. This involving integral and ordinary differential equation with constant and variable coefficient It has some applications in nearly all engineering disciplines, like System Modeling, Analysis of Electrical Circuit, Digital Signal Processing, Nuclear Physics, Process Controls, Applications in Probability, Applications in Physics, Applications in Power Systems Load Frequency Control, Mat lab etc.

Definition

Let F (t) is a well defined function of t for all t ≥ 0. The Laplace transformation of F (t), denoted by f (p) or L {F (t)}, is defined as

L {F (t)} $=\int_0^\infty e^{-pt}\,F(t)dt = f(p)$

Provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not. Where p the parameter which may be real or complex number and L is the Laplace transformation operator.

The Laplace transformation of F (t) i.e. $\int_0^\infty e^{-pt}\,F(t)dt$ exists for p>a, if

F (t) is continuous and $\lim_{n\to\infty}\{e^{-at}\,F(t)\}$ is finite. It should however, be keep in mind that above condition are sufficient and not necessary.

(A)

Simultaneous Linear Differential Equations:

$$\frac{dx}{dt} - y = e^{kt}$$

Or $x' - 11y = e^{kt}$ ……………..(1)

$$\frac{dy}{dt} + 23x = Sin4t$$

Or

$$y' + 23y = Sin4t \text{ …………… (2)}$$

Given that, $x(0) = k, y(0) = 0$.

Taking Laplace Transformations on both sides of (1),

$$L\{x'\} - L\{y\} = L\{e^{kt}\}$$

$$p\bar{x}(p) - x(0) - \bar{y}(p) = \frac{1}{p-k}$$

$$p\bar{x}(p) - \bar{y}(p) = \frac{1}{p-k} + k$$

$$p\bar{x}(p) - \bar{y}(p) = \frac{p}{p-k} \dots\dots\dots\dots\dots (3)$$

Taking Laplace Transformations on both sides of (1),
$L\{y'\} + 23L\{y\} = L[Sin4t\}$

$$p\bar{y}(p) - y(0) + 23\,\bar{x}(p) = \frac{4}{p^2+16}$$

$p\bar{y}(p) + 23\,\bar{x}(p) = \frac{4}{p^2+16}$..............(4)
Solving (3) & (4) we get,

$$(p^2+23)\bar{x}(p) = \frac{p^2}{p-k} + \frac{4}{p^2+16}$$

$$\bar{x}(p) = \frac{p^2}{(p-k)(p^2+23)}\ (I) + \frac{4}{(p^2+16)(p^2+23)}\ (II)$$

Now, Taking (I), $\frac{p^2}{(p-k)(p^2+23)}$
By partial fractions Method,

$$\bar{x}(p) = \frac{k^2}{(p-k)(k^2+23)} + \frac{23p}{(k^2+23)(p^2+23)} + \frac{23k}{(k^2+23)(p^2+23)}$$

Taking inverse Laplace Transformations on both sides

$$x(t) = \frac{k^2}{(k^2+23)}L^{-1}\left\{\frac{1}{p-k}\right\} + \frac{23}{(k^2+23)}L^{-1}\left\{\frac{p}{(p^2+23)}\right\} + \frac{23k}{(k^2+23)}L^{-1}\left\{\frac{1}{(p^2+23)}\right\}$$

Hence,

$$x(t) = \frac{k^2}{(k^2+23)}e^{kt} + \frac{23}{(k^2+23)}Cos\sqrt{23}t + \frac{k\sqrt{23}}{(k^2+23)}Sin\sqrt{23}t$$

Taking (II), $\frac{4}{(p^2+16)(p^2+23)}$
By partial fractions Method,

$$\bar{x}(p) = \frac{4}{(p-23)(p^2+16)}$$

$$\bar{x}(p) = \frac{4}{7(p^2+23)} - \frac{4}{7(p^2+16)}$$

Taking inverse Laplace Transformations on both sides

$$x(t) = \frac{4}{7}L^{-1}\left\{\frac{1}{(p^2+23)}\right\} - \frac{4}{7}L^{-1}\left\{\frac{1}{(p^2+16)}\right\}$$

hence,

$$x(t) = \frac{4}{7\sqrt{23}}Sin\sqrt{23}\,t - \frac{1}{7}Sin4t$$

So,

$$x(t) = \frac{k^2}{(k^2+23)}e^{kt} + \frac{23}{(k^2+23)}Cos\sqrt{23}t + \frac{k\sqrt{23}}{(k^2+23)}Sin\sqrt{23}t + \frac{4}{7\sqrt{23}}Sin\sqrt{23}\,t - \frac{1}{7}Sin4t$$

Again, [For value of y] Solving (3) & (4) we get,

$$\bar{y}(p) = \frac{4}{(p-23)(p^2+16)}\ (III) - \frac{23}{(p-23)\,(p-k)}\ (IV)$$

Taking (III),

$$\bar{y}(p) = \frac{4}{(p-23)(p^2+16)}$$

By partial fractions Method,

$$\bar{y}(p) = -\frac{4p}{545(p^2+16)} + \frac{92}{545(p^2+16)} + \frac{1}{545(p-23)}$$

Taking inverse Laplace Transformations on both sides,

$$y(t) = -\frac{4}{545}L^{-1}\left\{\frac{p}{(p^2+16)}\right\} - \frac{92}{545}L^{-1}\left\{\frac{1}{(p^2+16)}\right\} + \frac{4}{545}L^{-1}\left\{\frac{p}{(p-23)}\right\}$$

$$y(t) = -\frac{4}{545}Cos4t - \frac{92}{2180}Sin4t + \frac{4}{545}e^{23t}$$

And taking (IV),

$$\bar{y}(p) = -\frac{23p}{(p-23)\,(p-k)}$$

By partial fractions Method,

$$\bar{y}(p) = \frac{23k}{(k-23)\,(p-k)} - \frac{529k}{(k-23)\,(p-23)}$$

Taking inverse Laplace Transformations on both sides,

$$y(t) = \frac{23k}{k-23}L^{-1}\left\{\frac{1}{p-k}\right\} - \frac{529k}{k-23}L^{-1}\left\{\frac{1}{p-23}\right\}$$

So,

$$y(t) = \frac{23k}{k-23}e^{kt} - \frac{529k}{k-23}e^{23t}$$

Hence,

$$y(t) = -\frac{1}{545}Cos4t - \frac{23}{2180}Sin4t + \frac{1}{545}e^{23t} + \frac{23k}{k-23}e^{kt} - \frac{529k}{k-23}e^{23t}$$

$$(\boldsymbol{B})\ \frac{dx}{dt} - \frac{dy}{dt} - 2x + 2y = 1 - 2t$$

$Or,\ x' - y' - 2x + 2y = 1 - 2t$….. (1)

$$\frac{d^2x}{dt^2} + 2\frac{dy}{dt} + x = 0$$

$Or,\ x'' + 2y' + x = 0$…………….. (2)

Given that, $x(0) = 0, y(0) = 0$ and $x'(0) = 0$.

Taking Laplace Transformations on both sides of (1),

$L\{x'\} - L\{y'\} - 2L\{x\} + 2L\{y\} = L\{1 - 2t\}$

$$p\bar{x}(p) - x(0) - p\bar{y}(p) + y(0) - 2\bar{x}(p) + 2\bar{y}(p) = \frac{1}{p} - \frac{2}{p^2}$$

$$p\bar{x}(p) - p\bar{y}(p) - 2\bar{x}(p) + 2\bar{y}(p) = \frac{1}{p} - \frac{2}{p^2}$$

$$(p-2)\bar{x}(p) - (p-2)\bar{y}(p) = \frac{1}{p} - \frac{2}{p^2} \quad \ldots\ldots\ldots (3)$$

Taking Laplace Transformations on both sides of (2),

$L\{x''\} + 2L\{y'\} + L\{x\} = 0$

$$p^2\bar{x}(p) - px(0) - x'(0) + 2p\bar{y}(p) - 2y(0) + \bar{x}(p) = 0$$

$$p^2\bar{x}(p) + 2p\bar{y}(p) + \bar{x}(p) = 0$$

$$(p^2+1)\bar{x}(p) + 2p\bar{y}(p) = 0 \ldots\ldots\ldots\ldots (4)$$

Solving (3) & (4),

$$\bar{x}(p) = \frac{2}{(p-2)(p+1)^2} - \frac{4}{p(p-2)(p+1)^2}$$

$$\bar{x}(p) = \frac{2p-4}{p(p-2)(p+1)^2}$$

By partial fractions Method,

$$\bar{x}(p) = \frac{2}{p} - \frac{2}{p+1} - \frac{2}{(p+1)^2}$$

$$\textit{Taking inverse Laplace Transformations on both sides,}$$

$$x(t) = 2L^{-1}\left\{\frac{1}{p}\right\} - 2L^{-1}\left\{\frac{1}{p+1}\right\} - 2L^{-1}\left\{\frac{1}{(p+1)^2}\right\}$$

$$x(t) = 2 - 2e^{-t} - 2te^{-t}$$

$$x(t) = 2(1 - e^{-t} - te^{-t})$$

Again, Solving, [For value of y] (3) & (4),

$$\bar{y}(p) = -\frac{(p-2)(p^2+1)}{p^2(p-2)(p+1)^2}$$

By partial fractions Method,

$$\bar{y}(p) = -\left[\frac{-2p+1}{p^2} + \frac{2p+2}{(p+1)^2}\right]$$

$$\textit{Taking inverse Laplace Transformations on both sides,}$$

$$y(t) = -\left[-2L^{-1}\left\{\frac{1}{p}\right\} + L^{-1}\left\{\frac{1}{p^2}\right\} + 2L^{-1}\left\{\frac{p}{(p+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^2}\right\}\right]$$

$$y(t) = -[-2 + t + 2t + 2te^{-t}]$$

$$y(t) = [2 - t - 2e^{-t} - 2te^{-t}]$$

Conclusion:

The main purpose of this paper is to give a brief idea about applications of Laplace transforms in various areas and how to solve the Simultaneous Linear Differential Equations. The primary use of Laplace transformation is converting a time domain functions into frequency domain function. Laplace transformation is a very useful mathematical tool to make simpler complex problems in the area of stability and control.

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