Fuzzy differential equations solution approaches

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Abstract: In this paper, a discussion around the approaches of solving fuzzy differential equation. Then the linear transformation approach is introduced with examples by using Gazilov way as a first approach, then introduced the differential equation under generalized hukuhara differentiability as a second approach with numerical examples in order to compare the different approaches.
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1. Introduction:

Fuzzy differential equation (FDE) has been rapidly developing in recent years, and has attracted many researchers. The use of FDE is a smart way to model dynamic systems under uncertain information [25]. The nation of fuzzy derivative was first induced by (Zadeh and Chang) [9], it was followed up by (Dubois and Prade) [10], also other process has been discussed by (Puri and Ralescu and Goetschel and Voxman) [20, 12].

The concept of differential equation in fuzzy environment concepts was formulated by (Kaleva) [17], using of Hukuhara or generalized derivatives the solution turns fuzzier as time goes by [11]. But (Bede) found that a large class of BVPs has no solution if hukuhara derivative is used [3], so to overcome this, the concept of generalized derivative was developed [2, 7]. (Khastan and Nieto) found solutions of a large class of BVPs using the generalized derivative [18].

(Stefanini and Bede) by the concept of generalization of hukuhara difference for compact convex set and introduced generalized hukuhara differentiability for fuzzy valued function, the demonstrated that [2,23].

Recently, (Gasilov) solve the fuzzy initial value problem by a new technique (linear transformation) [13] and (Barros) solve fuzzy differential equation by fuzzification of the derivative operator [6].

Due to the wide applications of the second order fuzzy differential equation, it is considered as the most important between all fuzzy differential equations, So, Many researchers have worked on the second FDE, (Wang and Gue) [28] solve second order by Adomian, (Gasilov) [13, 14] solve by linear transformation, (Ahmadi) apply fuzzy Laplace transform [20], (jamoshidi and Avazpour) found way by shooting method [16], while (Rabiei) solved by improved runge kutta [22], Finally (Mondal and ray) solved in fuzzy environment analytically [26].

In this study, we investigate a differential equation with fuzzy boundary values. We define the problem as a set of crisp problems. For linear equations, we propose a method based on the properties of linear transformations. We show that, if the solution of the corresponding crisp problem exists and is unique, then the fuzzy problem also has unique solution.

Moreover, we prove that if the boundary values are triangular fuzzy numbers, then the value of the solution is a triangular fuzzy number at each time. We explain the proposed method on examples. We find analytical expression for solution of second-order linear differential equation with constant coefficients.

2. Basic concepts

In this section, we will illustrate the fundamental concepts and facts related to fuzzy differential equations. According to Zadeh [25], a fuzzy set is a generalization of a classical set that allows the membership function to take any value in the unit interval [0, 1].

Definition 2.1[1-4] Let U a nonempty universe and fuzzy set (A) in U is a function $A:U \to [0,1]$. Where $\mu(x)$ is the degree of membership of $x \ in \ A$, when $\mu(x)$ goes closer to 1, the x is more considered to belong to A, but when it goes closer to 0, the x is less considered to belong to A. $A = \{(x, \mu(x)), x \in X\}$

Definition 2.2 [1-4] Let A be a fuzzy set in U, the support of A is the crisp set in all elements in U with non-zero membership in A. $Supp(A) = \{x \in A | A(x) > 0\}$

Definition2.3 [1-4] Let A be a fuzzy set in U, the core of A is the crisp set in all elements in U with membership in A equals 1. $Core(A) = \{x \in A | A(x) = 1\}$

Definition 2.4[1-4] Let A be a fuzzy set in R. A Is called a fuzzy interval if:

(i) A is normal: there exists $x_0 \in R$ then $A(x_0) = 1$

(ii) A is Convex: for all $x, y \in R$ and $0 \le \lambda \le 1$, it holds that $A(\lambda x + (1-\lambda)y) \ge \min(A(x), A(y))$;

(iii) A is upper semi-continuous $A(x_0) \ge \lim_{x \to x_0^{\pm}} A(x)$;

(iv) $[A]^0 = \overline{supp(A)}$ is compact subset of R.

Definition 2.5: [12,23] Let *A* be a fuzzy set then, $\alpha - cut\ of\ A$ Is the crisp set of $[A]^\alpha$ that contains all elements with membership greater than or equal α. Where; $\alpha \in]0,1]$ $[A]^\alpha = \{x \in R | A(x) \ge \alpha\}$. $[A]^\alpha = [a_1^\alpha, a_2^\alpha]$. Where a_1^α is lower and a_2^α is upper.

Definition 2.6 [3,17]: Generalized triangle fuzzy number $GTFN$

We can say that this number is a special case of generalized trapezoidal fuzzy number when the core becomes point not interval.

$$\overline{A}_{GTFN} = (a_1, a_2, a_3, \omega);$$

$$\mu(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & if\ a_1 \le x \le a_2 \\ \omega & if\ a_2 = x \\ \omega \frac{a_3-x}{a_3-a_2} & if a_2 \le x \le a_3 \\ 0 & \end{cases};$$

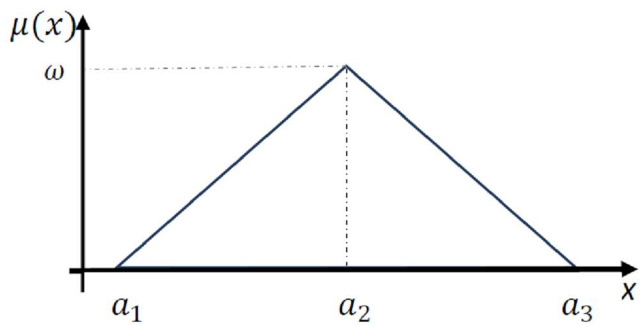

Fig. 1: triangle fuzzy number

At $\omega = 1$, we can get **triangle fuzzy number**.

Definition 2.7: generalized trapezoidal fuzzy number ($GTrFN$ **) [17]**

$\overline{A}_{GTFN}$ is a subset of IFN in R with following membership:

$$\overline{A}_{GTFN} = (a_1, a_2, a_3, a_4, \omega);$$

$$\mu(x) = \begin{cases} \omega \frac{x-a_1}{a_2-a_1} & if\ a_1 \le x \le a_2 \\ \omega & if\ a_2 \le x \le a_3 \\ \omega \frac{a_4-x}{a_4-a_3} & if a_3 \le x \le a_4 \\ 0 & otherwise \end{cases};$$

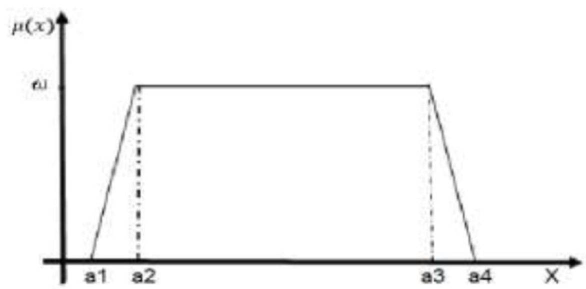

Fig. 2: $GTrFN$ representation

At $\omega = 1$, we can get **trapezoidal fuzzy number**.

Definition 2.8: [11], distance between two fuzzy intervals

Let *A* and *B* are two fuzzy intervals then, Hausedorff distance [11] between $[A]^\alpha$ and $[b]^\alpha$ is; $d_H([A]^\alpha, [B]^\alpha) = \max\{|a_1^\alpha - b_1^\alpha|, |a_2^\alpha - b_2^\alpha|\}$ By using Hausedorff distance, it is easily to find the distance between two fuzzy intervals which can be written as following $D(A, B) = sup_{\alpha \in [0,1]} d_H([A]^\alpha, [B]^\alpha)$.

Definition 2.9: [10], Let x, y *and* z are fuzzy numbers, and there exists $x = y + z$

Then, Hukuhara difference is $z = x \ominus y = [z_1^\alpha, z_2^\alpha] = [x_1^\alpha, x_2^\alpha] \ominus [y_1^\alpha, y_2^\alpha] = [x_1^\alpha - y_1^\alpha, x_2^\alpha - y_2^\alpha]$

, Where $x \ominus y \neq x + (-y)$.

Then, Generalized Hukuhara difference is $z = x \ominus_{gh} y = [z_1^\alpha, z_2^\alpha]$

$then;\ z_1^\alpha = \min[(x_1^\alpha - y_1^\alpha), (x_2^\alpha - y_2^\alpha)]$, $z_2^\alpha = \max[(x_1^\alpha - y_1^\alpha), (x_2^\alpha - y_2^\alpha)]$.

Definition 2.10: [4-7,23], The generalized hukuhara first derivative of a fuzzy parametric function is defined as; $f'(t_0) = \lim_{h\to 0} \frac{f(t_0+h) \ominus_{gh} f(t_0)}{h}$, From the definition, we have two classes:

(i) -differentiable at t_0 $[f'(t_0)]_\alpha = [f_1'(t_0,\alpha), f_2'(t_0,\alpha)]$

(ii) -differentiable at t_0 $[f'(t_0)]_\alpha = [f_2'(t_0,\alpha), f_1'(t_0,\alpha)]$

Where, f_1 is the lower and f_2 is the upper.

Definition 2.11: [23, 24], The generalized hukuhara second derivative of fuzzy function is defined as; $f''(t_0) = \lim_{h\to 0} \frac{f'(t_0+h) \ominus_{gh} f'(t_0)}{h}$

According to the Definition 12, we have the following classes:

$f'(t_0)$ is (i)--differentiable if:

$$f''(t_0) = \begin{cases} [f_1''(t_0,\alpha), f_2''(t_0,\alpha)]\ if\ f\ is\ (i) - differentiable \\ \qquad class(1,1) \\ [f_2''(t_0,\alpha), f_1''(t_0,\alpha)]\ if\ f\ is\ (ii) - differentiable \\ \qquad class(2,2) \end{cases}$$

$f'(t_0)$ Is (ii)-differentiable if:

$$f''(t_0) = \begin{cases} [f_2''(t_0,\alpha), f_1''(t_0,\alpha)]\ if\ f\ is\ (i) - differentiable \\ \qquad class(1,2) \\ [f_1''(t_0,\alpha), f_2''(t_0,\alpha)]\ if\ f\ is\ (ii) - differentiable \\ \qquad class(2,1) \end{cases}$$

Definition 2.12: [26] Let $[x_1(t,\alpha), x_2(t,\alpha)]$ be solution of any fuzzy differential is called a strong solution, if $\frac{dx_1(t,\alpha)}{d\alpha} > 0, \frac{dx_2(t,\alpha)}{d\alpha} < 0\ \forall \alpha \in [0,\omega], x_1(t) \le x_2(t)$

Otherwise it is called a weak solution.

3. Fuzzy boundary value problem in different approaches

Fuzzy differential equations play an important role in increasing number of system models in engineering, physics and other sciences. For example, civil engineering models like a queuing model for earthwork and a model of oscillations of bell-towers. in modelling hydraulic, Fuzzy differential equations in modelling hydraulic differential servo cylinders by Bede and Fodor. Also the use of fuzzy differential equations to model dynamic systems and Oscillatory problems under uncertainty conditions.

There are many approaches in solving the second order FDE. These are

a) The First approach is the method based on linear transform. Split up the problem into two parts, corresponding crisp problem and the fuzzy problems.

b) The second one is Hukuhara or generalized derivative. There is some difficulty in using Hukuhara derivative approach. To overcome the difficulty generalized derivative was developed.

c) The third approaches are extension principle. In this method, we solve the associated crisp differential equation and then fuzzify the solutions.

d) The another approaches is numerical solution of this FDE.

3.1. first approach

In this section, we are going to introduce the first approach the method based on linear transform. Split up the problem into two parts, corresponding crisp problem and the fuzzy problems. In some applications, the behaviour of an object can be determined by physics laws and these laws give crisp solution. However, if the boundary values are obtained by measurements with some errors, so these values are uncertain and it is more suitable to be modelled by fuzzy numbers and this gives rise to BVPs with crisp dynamics but with fuzzy boundary values.

consider the fuzzy boundary value problem with crisp linear differential equation but with fuzzy boundary condition.

$$x'' + a_1(t)x' + a_2(t)x = f(t) \qquad (3.1)$$
$$x(t_0) = \tilde{A} \qquad x(T) = \tilde{B}$$

According to the approach, we will rewrite the boundary condition $\tilde{A} = a_{cr} + \tilde{a}$ and $\tilde{B} = b_{cr} + \tilde{b}$, this way of writing the condition may be the meaning of transformation that we can see the uncertain parts have been moved to vertices at zero.

So, The problem is splited into two problems:

1) Associated crisp problem (certain

2) $x'' + a_1(t)x' + a_2(t)x = f(t)$

$x(t_0) = a_{cr} \qquad x(T) = b_{cr}$ (3.2)

3) Homogenous problem with fuzzy condition

$$x'' + a_1(t)x' + a_2(t)x = 0$$

$$x(t_0) = \tilde{a} \qquad x(T) = \tilde{b} \tag{3.3}$$

Easily it can be estimated that the final solution for equation (1) will be $x(t) = x_{cr}(t) + x_{un}(t)$, which are the solutions of (2) and (3), the solution of problem (2) is can prepared by many analytical methods, but it is different at problem (3), the solution $x_{un}(t)$ of problem (3) is assumed to be a fuzzy set $\tilde{x}$ of real function such as $x(t)$, each $x(t)$ must satisfy the differential equation and must have boundary conditions a and b from the fuzzy sets $\tilde{a}$ and $\tilde{b}$, where the membership of the solution at least meets the least membership of its boundary.

$$\tilde{x} = \{x(t)|x'' + a_1(t)x' + a_2(t)x = 0; x(t_0) = a; x(T) = b; a \in \tilde{a}; b \in \tilde{b}\} \tag{3.4}$$

With membership function

$$\mu_{\tilde{x}}(x(t)) = \min\{\mu_{\tilde{a}}(a), \mu_{\tilde{b}}(b)\} \tag{3.5}$$

Let us illustrate the solution methodology,

Here a crisp case of second order linear differential equation;

$$x'' + a_1(t)x' + a_2(t)x = 0$$

$$x(t_0) = a \qquad x(T) = b \tag{3.6}$$

Let x_1 and x_2 be linear independent solutions of the differential equation, Then the general solution $x(t) = c_1x_1 + c_2x_2$,

For evaluating the constants:

$$\begin{cases} c_1x_1(t_0) + c_2x_2(t_0) = a \\ c_1x_1(T) + c_2x_2(T) = b \end{cases} \tag{3.7}$$

This linear system in (7) can be represented in matrix form $Mc = u$, where $M = \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ x_1(T) & x_2(T) \end{bmatrix}$; $c = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$; $u = \begin{bmatrix} a \\ b \end{bmatrix}$.

Then the constant is

$$c = M^{-1}u \tag{3.8}$$

Let us consider $S(t) = [x_1(t) \quad x_2(t)]$ be a vector-function of linear independent solution, then the general solution is

$$x(t) = [x_1(t) \quad x_2(t)]\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = S(t).c \qquad 3.9)$$

From (8), $x(t) = S(t).M^{-1}u$, And Let $w(t) = S(t).M^{-1}$, then we get the final form of the general crisp solution.

$$x(t) = w(t)u = w_1(t)a + w_2(t)b \tag{3.10}$$

According to (10) and Linear transformation properties, we can say that the solution of the fuzzy boundary problem is

$$\tilde{x}(t) = w(t)\tilde{u} = w_1(t)\tilde{a} + w_2(t)\tilde{b} \tag{3.11}$$

Lemma 3.1. $\tilde{x}(t)$, the value of the solution of (2.20) at a given time, is well-defined as a fuzzy number.

Proof. According to (2.22), an $\alpha - cut$ of $\tilde{x}$ is expressed as

$$x_\alpha = \{x(t)|x'' + a_1(t)x' + a_2(t)x = 0; x(t_0) = a; x(T) = b; a \in [\underline{a_\alpha}, \overline{a_\alpha}]; b \in [\underline{b_\alpha}, \overline{b_\alpha}]\} \tag{2.29}$$

Let linear independent solutions of (2.6) are known then, according to (2.28) and (2.29).

$$x_\alpha = \{x(t)|x = w.u; u = (a,b): a \in [\underline{a_\alpha}, \overline{a_\alpha}]; b \in [\underline{b_\alpha}, \overline{b_\alpha}]\} \tag{3.12}$$

Consider a fixed time *t* and let $v = w(t)$, then

$$x_\alpha = \{v.u; u \in [\underline{a_\alpha}, \overline{a_\alpha}] \times [\underline{b_\alpha}, \overline{b_\alpha}]\}, \tag{3.13}$$

Consider the transformation $T(u) = v.u$ and x_α is the image of $k_\alpha = [\underline{a_\alpha}, \overline{a_\alpha}] \times [\underline{b_\alpha}, \overline{b_\alpha}]$ under linear transformation $T(u)$.

Let us, discuss the case of triangle boundary values. If $\tilde{a}$ and $\tilde{b}$ are triangular fuzzy numbers, the $\alpha - cuts$ of the region $\tilde{R} = (\tilde{a}, \tilde{b})$ are nested similar rectangles, their images are intervals that also are nested and similar, then $\tilde{X}(t) = (\underline{x_0}(t), 0, \overline{x_0}(t))$, to calculate $\underline{x_0}(t)$ and $\overline{x_0}(t)$.

Let $\tilde{a} = (\underline{a_0}, 0, \overline{a_0})$, $\tilde{b} = (\underline{b_0}, 0, \overline{b_0})$ and $w(t) = (w_1(t), w_2(t))$ Then

$$\overline{x_0}(t) = \max[\underline{a_0}w_1(t), \overline{a_0}w_1(t)] + \max[\underline{b_0}w_2(t), \overline{b_0}w_2(t)] \tag{3.14}$$

And,

$$\underline{x_0}(t) = \min[\underline{a_0}w_1(t), \overline{a_0}w_1(t)] + \min[\underline{b_0}w_2(t), \overline{b_0}w_2(t)] \tag{3.15}$$

Note also that

$$x_\alpha(t) = [\underline{x_\alpha}(t), \overline{x_\alpha}(t)] = (1-\alpha)x_0(t) \quad (3.16)$$

Then,

$$\tilde{X}(t) = w_1(t)\tilde{a} + w_2(t)\tilde{b} \quad (3.17)$$

3.2. Second approach

In this section, the study concerns with the fuzzy effect and using the generalized Hukuhara differentiability on the following FBVP [24]:

$$\tilde{y}''(x) = p\tilde{y}'(x) + q\tilde{y}(x) \quad (3.18)$$

According to the following fuzzy boundary conditions

$$\tilde{y}(x_0) = \tilde{a}, \qquad \tilde{y}(x_b) = \tilde{b}, \quad (3.19)$$

Where p and q are constants, $\tilde{a}$ and $\tilde{b}$ are two generalized trapezoidal fuzzy number represented as:

$\tilde{a} = (a_1, a_2, a_3, a_4; \omega)$ And $\tilde{b} = (b_1, b_2, b_3, b_4; \omega)$

The lower and upper values of $\tilde{a}$ and $\tilde{b}$ are given according to Table 1:

table 1: the lower and upper values

	Lower values	Upper values
$\tilde{a} = [\underline{a}, \overline{a}]_\alpha$	$\underline{a} = a_1 + \frac{\alpha(a_2 - a_1)}{\omega}$	$\overline{a} = a_4 - \frac{\alpha(a_4 - a_3)}{\omega}$
$\tilde{b} = [\underline{b}, \overline{b}]_\alpha$	$\underline{b} = b_1 + \frac{\alpha(b_2 - b_1)}{\omega}$	$\overline{b} = b_4 - \frac{\alpha(b_4 - b_3)}{\omega}$

Solution of fuzzy boundary value problems

Class (1, 1)

The FBVP (1) can be written as:

$$\overline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \quad (x_0,\alpha) = \overline{a}, \quad (x_b,\alpha) = \overline{b}, \quad (3.20)$$

And

$$\underline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \quad (x_0,\alpha) = \underline{a}, \quad (x_b,\alpha) = \underline{b} \quad (3.21)$$

.

In order to get the solution of (3, 4), we write these equations in the following system:

$$d_1 = p + \sqrt{p^2 + 4q}, \qquad d_2 = p + \sqrt{p^2 - 4q},$$

And the constants c_1, c_2, c_3 and c_4 can be obtained by applying the fuzzy boundary condition given by Eq. (3, 4).

By using the value of $c, \underline{a}, \overline{a}, \underline{b}$ and $\overline{b}$ then:

$$\underline{y}(x,\alpha) = \frac{\underline{a}(e^{\frac{d_1x_b+d_3x}{2}} - e^{\frac{d_3x_b+d_1x}{2}}) + \underline{b}(e^{\frac{d_3x_0+d_1x}{2}} - e^{\frac{d_1x_0+d_3x}{2}})}{(e^{\frac{d_1x_b+d_3x_0}{2}} - e^{\frac{d_3x_b+d_1x_0}{2}})} \quad (3.27)$$

$$\overline{y}(x,\alpha) = \frac{\overline{a}(e^{\frac{d_1x_b+d_3x}{2}} - e^{\frac{d_3x_b+d_1x}{2}}) + \overline{b}(e^{\frac{d_3x_0+d_1x}{2}} - e^{\frac{d_1x_0+d_3x}{2}})}{(e^{\frac{d_1x_b+d_3x_0}{2}} - e^{\frac{d_3x_b+d_1x_0}{2}})} \quad (3.28)$$

Where, $\underline{a}, \overline{a}, \underline{b}$ and $\overline{b}$ values from table 1.

Class (1, 2)

$$\begin{bmatrix} r \\ u \\ z \\ w \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ q & p & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & q & p \end{bmatrix} \begin{bmatrix} r \\ u \\ z \\ w \end{bmatrix}, \quad (3.24)$$

Where $\underline{y} = r$ and $\overline{y} = z$.

So the lower and upper solutions are given by:

$$\underline{y}(x,\alpha) = c_2 e^{\frac{d_1x}{2}} + c_4 e^{\frac{d_3x}{2}}, \quad (3.25)$$

$$\overline{y}(x,\alpha) = c_1 e^{\frac{d_1x}{2}} + c_3 e^{\frac{d_3x}{2}}, \quad (3.26)$$

Where, d_1, d_2, d_3 and d_4 are constants given by:

$$= p - \sqrt{p^2 + 4q}, \qquad = p - \sqrt{p^2 - 4q}$$

$$\underline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \overline{y}(x_0,\alpha) = \overline{a}, \quad (x_b,\alpha) = \overline{b}, \quad (3.29)$$

$$\overline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a}, \quad (x_b,\alpha) = \underline{b}, \quad (3.30)$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_2} + c_2 e^{\frac{-x}{2}d_4} + c_3 e^{\frac{x}{2}d_3} + c_4 e^{\frac{x}{2}d_1}, \quad (3.31)$$

$$\overline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_2}\left(1 - d_4(p/q)\right) + c_2 e^{\frac{-x}{2}d_4}\left(1 - d_2(p/q)\right) + c_3 e^{\frac{x}{2}d_3} + c_4 e^{\frac{x}{2}d_1} \quad (3.32)$$

Class (2, 1)

$$\overline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \overline{y}(x_0,\alpha) = \overline{a}, \overline{y}(x_b,\alpha) = \overline{b}, \quad (3.33)$$

$$\underline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a}, \underline{y}(x_b,\alpha) = \underline{b}, \quad (3.34)$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{x}{2}d_2} + c_2 e^{\frac{x}{2}d_1} + c_3 e^{\frac{x}{2}d_4} + c_4 e^{\frac{x}{2}d_3} \quad (3.35)$$

$$\overline{y}(x,\alpha) = -c_1 e^{\frac{x}{2}d_2} + c_2 e^{\frac{x}{2}d_1} - c_3 e^{\frac{x}{2}d_4} + c_4 e^{\frac{x}{2}d_3} \quad (3.36)$$

Class (2, 2)

$$\overline{y}''(x,\alpha) = p.\underline{y}'(x,\alpha) + q.\overline{y}(x,\alpha), \overline{y}(x_0,\alpha) = \overline{a}, \overline{y}(x_b,\alpha) = \overline{b}, \quad (3.37)$$

$$\underline{y}''(x,\alpha) = p.\overline{y}'(x,\alpha) + q.\underline{y}(x,\alpha), \underline{y}(x_0,\alpha) = \underline{a}, \underline{y}(x_b,\alpha) = \underline{b}, \quad (3.38)$$

The general solution:

$$\underline{y}(x,\alpha) = c_1 e^{\frac{-x}{2}d_3} + c_2 e^{\frac{x}{2}d_3} + c_3 e^{\frac{-x}{2}d_1} + c_4 e^{\frac{x}{2}d_1}, \quad (3.39)$$

$$\overline{y}(x,\alpha) = -c_1 e^{\frac{-x}{2}d_3} + c_2 e^{\frac{x}{2}d_3} - c_3 e^{\frac{-x}{2}d_1} + c_4 e^{\frac{x}{2}d_1}, \quad (3.40)$$

4. Applications

Example 4.1.

Consider the following FBVP

$$\tilde{y}''(t) = 5\tilde{y}'(t) + 4\tilde{y}(t), \ \tilde{y}(0) = (0.8, 1, 1.1, 1.3; 0.7), \ \tilde{y}(1) = (2.6, 2.8, 3, 3.4; 0.7)$$

.

Class (1, 1)

The general solution is given by

$$\underline{y}(t,\alpha) = (0.48\alpha + 7.3)10^{-3} e^{\frac{(5+\sqrt{41})t}{2}} + (0.285\alpha + 0.79) e^{\frac{(5-\sqrt{41})t}{2}}$$

,

$$\overline{y}(t,\alpha) = (9.2 - 1.4\alpha)10^{-3} e^{\frac{(5+\sqrt{41})t}{2}} + (1.29 - 0.284\alpha) e^{\frac{(5-\sqrt{41})t}{2}}.$$

Also, for different values of α, we plotted the lower and upper solutions in Fig. 3, and we listed the lower and upper solutions for $t = 0.5$ in Table 2.

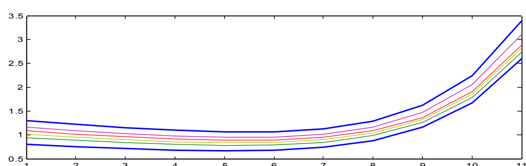

Fig. 3: The lower and upper solution class (1,1) of Example 1for $\alpha = 0$,0.5,0.7

Class (1, 2)

The general solution is given by

$$\underline{y}(t,\alpha) = (0.7 - 0.74\alpha)e^{-4t} + (0.242\alpha - 0.226)e^{-t} + (0.78\alpha + 0.312)e^{\frac{(5-\sqrt{41})t}{2}} + (0.008 - 0.0059\alpha)e^{\frac{(5+\sqrt{41})t}{2}}$$

,

$$\overline{y}(t,\alpha) = (0.7 - 0.74\alpha)e^{-4t}\left(-\left(3/2\right)\right) + (0.242\alpha - 0.226)e^{-t}(-9) + (0.78\alpha + 0.312)e^{\frac{(5-\sqrt{41})t}{2}} + (0.008 - 0.0059\alpha)e^{\frac{(5+\sqrt{41})t}{2}}$$

.

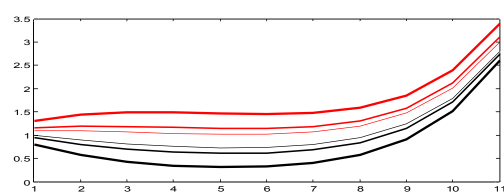

Fig. 4: The lower and upper solution class (1,2) of Example 1for $\alpha = 0$,0.5,0.7

Class (2, 1)

The general solution is given by

$$\underline{y}(t,\alpha) = (0.0054 - 0.0067\alpha)e^{4t} + (0.0083 - 0.00047\alpha)e^{\frac{(5+\sqrt{41})t}{2}} + (0.29\alpha - 0.255)e^{t} + (0.00047\alpha + 1.04)e^{\frac{(5-\sqrt{41})t}{2}},$$

$$\bar{y}(t,\alpha) = -(0.0054 - 0.0067\alpha)e^{4t} + (0.0083 - 0.00047\alpha)e^{\frac{(5+\sqrt{41})t}{2}}$$
$$-(0.29\alpha - 0.255)e^{t} + (0.00047\alpha + 1.04)e^{\frac{(5-\sqrt{41})t}{2}}$$

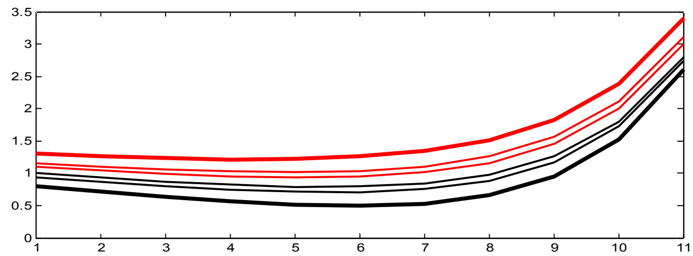

Fig. 5: The lower and upper solution class (2,1) of Example 1for $\alpha = 0$,0.5,0.7

Class (2, 2)

The general solution is given by

$$\underline{y}(t,\alpha) = (-0.071\alpha - 0.198)e^{\frac{-(5-\sqrt{41})t}{2}} + (1.042 - 0.0014\alpha)e^{\frac{(5-\sqrt{41})t}{2}}$$
$$+(0.357\alpha - 0.0517)e^{\frac{-(5+\sqrt{41})t}{2}} +$$
$$(0.0014\alpha + 0.0083)e^{\frac{(5+\sqrt{41})t}{2}},$$

and

$$\bar{y}(t,\alpha) = (0.071\alpha + 0.198)e^{\frac{-(5-\sqrt{41})t}{2}} + (1.042 - 0.0014\alpha)e^{\frac{(5-\sqrt{41})t}{2}}$$
$$-(0.357\alpha - 0.0517)e^{\frac{-(5+\sqrt{41})t}{2}} + (0.0014\alpha + 0.0083)e^{\frac{(5+\sqrt{41})t}{2}}$$

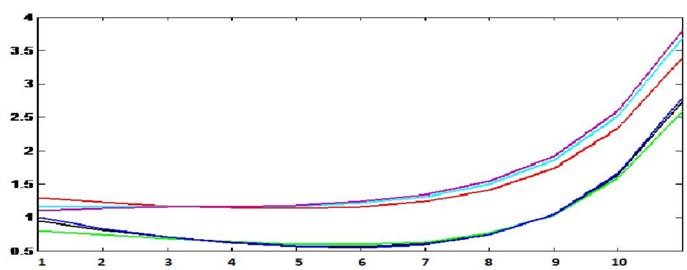

Fig. 6: The lower and upper solution class (2,2) of Example 1for $\alpha = 0$,0.5,0.7

From Fig. 6, we notice the intersection between the solution and this indicates that this class represents a weak solution which means that the inner band may become outer band at different t so, we can check the variation with α in Table 2.

Table 2: The lower and upper solutions of Example 1 for $t = 0.5$

	Class (1, 1)		Class (1, 2)		Class (2, 1)		Class (2, 2)	
α	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$	$\underline{y}(t,\alpha)$	$\bar{y}(t,\alpha)$
0	0.6857	1.0684	0.3239	1.4576	0.4958	1.2583	0.5925	1.1616
0.1	0.7066	1.0459	0.3827	1.3946	0.5383	1.2142	0.5868	1.1720
0.2	0.7275	1.0234	0.4416	1.3316	0.5807	1.1702	0.5811	1.1825
0.3	0.7484	1.0009	0.5005	1.2686	0.6232	1.2610	0.5754	1.1930
0.4	0.7694	0.9784	0.5594	1.2056	0.6656	1.0821	0.5697	1.2034
0.5	0.7903	0.9559	0.6182	1.1426	0.7081	1.0381	0.5640	1.2139
0.6	0.8112	0.9334	0.6771	1.0796	0.7506	0.9940	0.5583	1.2243
0.7	0.8321	0.9109	0.7360	1.0166	0.7930	0.9500	0.5526	1.2348

From Table 2, we conclude that $\underline{y}(t,\alpha)$ is increasing and $\overline{y}(t,\alpha)$ is decreasing

Then, the solution is a strong solution for class (1,1), (1,2) and (2,1) while the solution of class (2,2) is a weak solution

Example 4.2. Solve the following FBVP

$$x'' - 4x' + 4x = 1 - 2t^2$$

$$x(0) = (2,3,4) \qquad x(1) = (1,2,2.5)$$

Solution; we represent the solution $\tilde{x}(t) = x_{cr}(t) + \tilde{x}_{un}(t)$

1) The Solution $x_{cr}(t)$ of the crisp non-homogenous problem

2) $x'' - 4x' + 4x = 1 - 2t^2$

$$x(0) = 3 \qquad x(1) = 2$$

Then,

$x_{cr}(t) = \frac{-1}{2}(t+1)^2 + 3.5(1-t)e^{2t} + 4te^{2(t-1)}]$

3) The solution $\tilde{x}_{un}(t)$ of the fuzzy homogenous problem:

$$x'' - 4x' + 4x = 0$$

$$x(0) = (-1,0,1) \qquad x(1) = (-1,0,0.5)$$

Then, $S(t) = [x_1(t) \quad x_2(t)] = [e^{2t} \quad te^{2t}]$,

$$M = \begin{bmatrix} x_1(t_0) & x_2(t_0) \\ x_1(T) & x_2(T) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ e^2 & e^2 \end{bmatrix},$$

$w(t) = S(t).M^{-1} =$

$[((1-t)e^{2t}) \qquad te^{2(t-1)}]$.

Then, $\tilde{x}_{un}(t) = w_1(t)\,\tilde{a} + w_2(t)\,\tilde{b}$

$=$

$[(1-t)e^{2t}(-1,0,1) + (te^{2(t-1)})(-1,0,0.5)]$.

4) The Final solution $\tilde{x}(t) = x_{cr}(t) + \tilde{x}_{un}(t)$

In this example, the final solution will be represented by using $\alpha - cut$ levels and this representation is very important because it can give more data about the solution in different membership.

we can get

$$\underline{x_0}(t) = (t-1)e^{2t} - te^{2(t-1)}$$

$$\overline{x_0}(t) = (1-t)e^{2t} + 0.5te^{2(t-1)}$$

Since $x_\alpha(t) = (1-\alpha)\left[\underline{x_0}(t), \overline{x_0}(t)\right]$

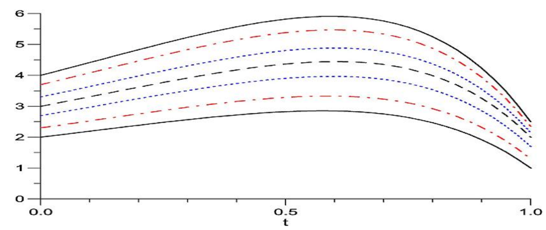

Figure (7) is the $\alpha - cut$ levels solution of example4.2

As it shown in figure 7, the solution of the problem in example 4.2. and this figure is diagram of $x_\alpha(t)$ with different levels of α the black dashed line is the crisp solution or the solution at $\alpha = 1$ and the blue dashed lines are the solution at $\alpha = 0.7$ while the red dashed line at $\alpha = 0.3$ banded with the upper and lower solid black lines and can be called 0-cuts.

Conclusion:

In this paper; the analytical solution of a second order differential equation with fuzzy conditions by using Gazilov way in linear transformation approach. From the results we confirmed that the solution existence and uniqueness due to the uniqueness of the crisp solution of the corresponding crisp problem. And also the analytical solution under generalized hukuhara differentiability and its advantage in solution and more data with more classes but the disadvantage in the switching points, a comparison between the two different approaches expand the way of research in this point.

So, in the future research we will solve the same problem but with adding a fuzzy non homogeneous term.

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