**Properties Standard Frame in Hilbert C\*- Modules**

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**Abstract:** The goal of the present paper is a short introduction to a general module frame theory in Hilbert C\* - modules over a unital C\*- algebra. In this paper firstly we recall some basic propertics of Hilbert space, Hilbert modules and modular standard frames then by using adjointable module homomorphism on Hilbert C\*- modules and on *l2(A)*. we construct some frames. Finally we present a relation between standard frame in Hilbert C\*- modules. We also study the behavior of Bessel sequences and frames under operators. In addition, we obtain a relation between standard frames in Hilbert C\* - modules. We focus on finitely and countably generated Hilbert *A*-module over unital C\* - algebra *A* and Our references are [1] and [6].

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**1. Introduction**

Hilbert space frames were originally introduced by Duffin and Schaffer[2] to deal with some problems in non-harmonic Fourier analysis. Hilbert  modules are generalizations of Hilbert spaces by allowing the inner product to take values in a  algebra rather than in the field of complex numbers [4]. These frames are called *Hilbert*  *modular frames* or just simply *modular frames*. These concepts are generalizations of some results in [11].

In this paper firstly we recall some basic propertics in Hilbert  modules, secondly by using adjointable module homomorphism on Hilbert  modules and onwe construct some frames and finally we present a relation between standara frames in Hilbert  modules. Our references for Hilbert space are [1] and [6].

**2.Perliminaries**

We review some basics about Hilbert  modules and Hilbert  modular frames.

For basic notaions and theory for Hilbert modules see[5,7,9,11].

In this paperwill denote the set of natural numbers and  will be a finite or countable Subset

of .

**Definition 2.1.** Let *A* be a (unital) algebra and *H*  be a (left)*A*-module. Suppose that The linear

structures given on *A* and *H* are compatible, i.e.  for every  and. Assume that there exists a mapping  with the properties:

 for every ,

 if and only if ,

 for every ,

for every , and every  ,

 for every .

Then the pair  is called a (left)pre-Hilbert *A*-module. The map  is said to be an *A*-valued inner product. If the pre-Hilbert *A*-module  is complete with respect to the induced norm  then it is called a Hilbert *A*-module.

In this paper we focus on finitely and countably generated Hilbert *A*-module over unital algebra *A*. In case *A* is unital the Hilbert *A*-module *H*  is *(algebraically) finitely generated* if there exists a finite set such that every element  can be expressed as an *A*-linear combination . A Hilbert *A*-module *H*  is *countably generated* if there exists a countable set  such that the set of all finite *A*-linear combinations, is norm-dense in *H*.

**Definition 2.2.**(see [4])Let *A* be a unital algebra and *J* be a finite or countable index set. A (finite or countable) sequence  of elements in a Hilbert *A*-module *H*  is said to be a *frame* for *H* if there exist two constants  such that for every 



If for every , the series in the middle of the inequality (1) is convergent in norm, we say that the frame is standard. The numbers *C* and *D* are called *frame bounds*. Likewise, sequence is called a (standard) *Bessel* *sequence* with Bessel bound D if there exists *D*>0 such that



The sequence satisfies the lower frame bound if there exists a *C*>0



The frame is saide to be a *tight frame* if *C=D*, and said to be *normalized* if *C=D*=1.

We consider standard (normalized tight) frames on finitely or countably generated Hilbert *A*-module over unital algebra *A.* For a unital algebra *A,*  let  be the Hilbert *A*-module, see[4], define by



For any standard frame  of a finitely or countably generated Hilbert *A*-module *H,* the frame transform of the frame is definded to be the map

that is bounded, *A*-linear and adjointable with adjoint



for a standard basis of the Hilbert *A*-module and all . See[5] Moreover for

every ,



Therefore  is one-to-one with a closed range which is complemented in , . We note that  is an invertible operator and the frame operator  is a positive invertible bounded operator on *H* such that for every ,

 The sequence is a frame for *H* and is called the *canonical dua l frame* of .

Now suppose that ****is a Bessel sequence of a finitely or countably generated Hilbert *A*-module *H,* the associated *analysis operator*  is defined by



Note that analysis operator is adjointable and adjoint  fulfills for all *j.*

Throughout this paper, we denote by , the set of all adjointable maps from *H* to *K* and as usual we abbreviate  by .

**3.Construction of frames in Hilbert module**

**Lemma 3.1.** Let *H* and *K* be Hilbert modules.

(*i*) If  is a Bessel sequence in *H* with bound *D* and  , then  is a Bessel sequence in *K* with bound ,

*(ii*) If  satisfies the lower frame condition and there exists a positive constant *B* such that for every ,  , then  satisfies the lower frame condition in .

**Proof.** (*i*) By proposition 1.2 of [8] for every *y*∈*K* we have



*(ii*) For every  we have



In the following theorem we give a necessary and suffient condition for  to be standard frame of  .

**Theorem 3.2.** Let be a standard frame of *H* with bounds 0<C≤D and T be a module map in , Then the statements are equinalent.

(*i*) The sequence  is a standard frame of  ;

*(ii*) There exists a positive costant *B* such that  satisfies:

(3)

For every .

**Proof.** Suppose thatis a standard frame of  with lower bound .

Then for every ,

(4) 

From which, condition (3) follow with .

The converse follows from the above lemma. Moreover since is a standard frame

 is convergent in norm, so is convergent in norm for every .

In previous theorem if  is a standard frame in *K*, then by the reconstruction formula (2),  is dense in *K*, so for to be a standard frame of *K* it is necessary than  be dense in *K* and consequently the assumption  yields the following result.

**Corollary 3.3.** Let be a standard frame of *H* with bounds 0<C≤D and T be a module map in such that . Then the statements are equinalent.

(*i*) The sequence is a standard frame of *K* ;

*(ii*) There exists a positive costant *B* such that  satisfies:

(5) 

for every .

**Remark.** (*a*)There exists a Hilbert *A*- module *H* and a module map ****suchthat  is injective, but  .(*cf.*[10]*, Exercise* 15.F). For this reason, in corollary (3.3) we supposed that . But if is a complemented submodule of *H* then condition (5) implise that .

(*b*) If *T* is a self adjoint module map in ****and satisfies condition (5), then *T* is invertible(*cf*. [7], *Lemma* 3.1). In particular *T* is surjective. Then  .

(*c*) Suppose that **** and  is closed. Then  is a complemented submodule of *H* and (*cf*. [8], *Theorem* 3.2). if satisfies condition (5), then  and .

**Corollary 3.4.** Let be a standard frame of *H.* If  *T* is an adjointable module map from *H* onto *K*, then the statements are equivalent.

(*i*) The sequence  is a standard frame of *K* ;

*(ii*) There exists a positive costant *B* such that  satisfies:

6)

for every .

By Corollary 3.4, we can construct some standard frames for a closed submodule of *H*, with a given standard frame.

Now, let  and let  be a standard frame of *H* with bounds  and and frame transform . We use *T* to construct the sequence  , where

(7) 

Such that



then



But the sequence  is not always a standard frame for *H* (e.g. T=0).

Now we want to make  a standard frame under an appropriate condition on *T*.

**Theorem 3.5.** Let be a standard frame of *H* with bounds ****and  **.** If **** then the following statements are equivalent.

(*i*) The sequence  is a standard frame of *H* defined by (7) ;

*(ii*) There exists a positive costant *B* such that  satisfies:



for every  where is the frame transform of  .

**Proof.** Suppose thatis a standard frame for *H* defined by (7). Then there are constants such that for every  ,

(8) 

Where is the frame transform of . Also for every  and ,



Since the set of *A*-linear combinations of  is dense in , we have

(9) 

So, by using the left inequality of (8), and (9), we conclude that



for every . Then



for every . Therefore sufficient we take .

Conversely, by using Proposition 1.2 of [8], since for every , is convergent in *A*, we have for every  ,

There fore is a frame with frame transform .

**4. Relation between standard frames in Hilbert *A*-module**

The aim of this section is to characterize all standard frames of *H*. In theorem 4.2, we will show how any two standard frames of H are related with each other.

**Definition 4.1**. Frames and of *H* and *K*, respectively, are similar if there is an *A*-linear adjointable, bounded operator such that for each  , and *T* is invertible.

**Theorem 4.2.** Let and be a standard frames in *H* and *K*, respectively, then they are similar. Conversely, if is a standard frame in *H* and is a frame in *K* which is similar to , then also is standard.

**Proof.** If and are transforms frames for and**,** respectively, then and are complemented in . Therefore the orthogonal projections  and  are adjointable. If we take , then *T* is an *A*-linear, bounded adoptable operator with such that for each,  and similarly the map is adjoitable with **** such that for each  , . Hence and  . Therefore we have the result. The converse is obvious.

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