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MARSLAND PRESS
Multidisciplinary Academic Journal Publisher

Fundamental of Ring Theory

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Abstract: Ring Theory provides information pertinent to the aspects of Ring Theory. This research paper covers a variety of topics related to ring theory, including restricted Zero Divisor, integral Domain, Division Ring or Skew Field, Boolean Ring, Nilpotent Ring, Sub Ring, Improper/Trivial Sub Ring, Homomorphism, Monomorphism, Epimorphism, Isomorphism, Auto Morphism, Kernal, center of Ring, Ideal, Quotient Ring, Maximal ideal, Principal ideal, Prime ideal and their Application. Semi-primary rings, finite free resolutions, generalized rational identities, quotient rings, idealizer rings, identities of Azumaya algebras, endomorphism rings, and some remarks on rings with solvable units. Organized into 35 pages, this research paper begins with an overview of the characterization of restricted Homomorphism, Kernal, Fundamental Theorems of Ring Homomorphism and their Application.
[Samia Arshad. **Fundamental of Ring Theory.** *Nat Sci* 2021;19(3):9-13]. ISSN 1545-0740 (print); ISSN 2375-7167 (online). http://www.sciencepub.net/nature. 2. doi:10.7537/marsnsj190321.02.

Key words: Homomorphism, Field, Ring, Fundamental of Ring Theory, Homomorphism theorem.

1. Introduction

In abstract algebra, Ring Theory is the study of rings algebraic structures in which addition and multiplication are defined and have similar properties to those operations defined for the integers. Ring theory studies the structure of rings, their representations, or, in different language, modules, special classes of rings (group rings, division rings, universal enveloping algebras), as well as an array of properties that provide to be of interest both within the theory itself and for its applications, such as homological properties and polynomial identities. Briefly, a ring is an abelian group with a second binary operation that is associative, is distributive over the abelian group operation and has an identity element. The abelian group operation is called addition and the second binary operation is called multiplication by extension from the integers. A familiar example of a ring is the integers. The integers form a commutative, since the order in which a pair of elements are multiplied does not change the result. The set of polynomials also forms a commutative ring with the usual operations of addition and multiplication of functions. An example of a ring that is not commutative is the ring of n×n real square matrices with n>= 2. Finally, a field is a commutative ring in which one can divide by any nonzero element: an example is the field of real members. Non commutative rings are quite different in flavor, since more unusual behavior can arise. While the theory has developed in its own right, a fairly recent trend has sought to parallel the commutative development by building the theory of certain classes of non commutative rings in a geometric fashion as if they were rings of functions on (non-existent) ' non commutative spaces'. This trend started in the 1980s with the development of non commutative geometry and with the discovery of quantum groups. It has led to a better understanding of non commutative rings, especially non commutative Noetherian rings. (Goodearl 1989)

3. Prelimineries Set:

A well defined collection of distinct objects.

Subset:

If every member of the set A is also a member of the set B, then A is said to be a subset of B.

Symbolically this is written as: $A \subseteq B$ (A is subset of B)

The relationship between sets established by $\subseteq$ is called inclusion or containment.

Proper Subset:

If A is a subset of B and B contains at least one element which is not an element of A, then A is said to be a proper Subset of B. In such case we write $A \subset B$ (A is proper Subset of B)

Improper Subset:

If A is a subset of B and A=B, then we say that A is an improper Subset of B. Every set A is an improper Subset of itself.

Equal Sets:

Two sets A and B are said to be equal if every element of A is contained in B and then vice versa. Symbolically A=B

Singleton Set:

A set containing only one element is called Singleton set. It is denoted by {1}

Power Set:

A collection of all subsets of a given set X is called power Set of X and denoted by P (X).

Binary Operation:

In mathematics a binary Operation is a calculation involving two input quantities. Binary Operations on sets are union, intersection and difference.

4. Group

Let G be a non-empty set and * be a binary Operation on G then G is called group if it satisfies the following actions.

I. Closure Law

$a*b \in G$ for all $a,b \in G$

II. Associative Law

$a*b \in G$ for all $a,b \in$ **G**

III. Identity Element

$\exists$ an element " e" in G such that

$a*e = e*a = a$ for all $a \in G$

IV. Inverse Element

For each $a \in G \; \exists \; a' \in G$ such that

$a*a' = a' * a = e$

Then G is called a group under * written as (G, *)

5.Abelian Group

If G is a group and $a*b = b*a$ for all $a,b \in G$. Then G is called an abelian group.

6. Ring Theory

A non empty set **R** together with two binary Operations namely addition and multiplication is said to be a ring if

(1) (**R**, +) is an Abelian group.

(2) (**R**,.) is a semi group.

(3) Left and right distribution laws are satisfied.

$a\,(a+b) = ab + ac$ for all $a,b,c \in \mathbf{R}$

$(a+b)c = ac + ab$ for all $a,b,c \in \mathbf{R}$

7. Field

A non empty set containing at least two elements with two binary Operation namely addition and multiplication is said to be field if it satisfies the following axioms.

(1) (**R,** +) is an abelian group.

(2) (**R** /{0},.) is an abelian group.

(3) Left and right distribution laws are satisfied.

Remarks: Every field is a ring but converse is not true in general.

Zero Divisor

If the product of two non zero elements in a ring is zero, then it is called zero divisor.

Z = { 0,1,2,3}

Integral Domain

A commutative ring with no zero divisor is called integral Domain.

$a.b = b.a$

Division Ring or Skew Field

A ring whose non zero elements from a group under multiplication.

" OR "

A commutative division ring is a field.

8. Boolean Ring

A ring **R** is said to be a Boolean ring if its every element of **R** is idempotent i.e. x^2 = x for all $x \in \mathbf{R}$

9. Nilpotent Ring

A ring for which a^n = 0 for all $a \in \mathbf{R}$ is called a nilpotent ring.

10. Sub Ring

A non empty subset **S** of a ring **R** is said to be a sub group of **R** if it is a ring itself.

Improper/ Trivial Sub Ring

If **R** is a ring then {0} and **R** are called improper or trivial sub groups.

11. Homomorphism

Let R and R' be two rings, a mapping $\phi: R \to R'$ is said to be homomorphism if

(I) $\phi(a+b) = \phi(a) + \phi(b)$

(II) $\phi(ab) = \phi(a) \bullet \phi(b)$

Monomorphism

A one-one homomorphism is called monomorphism.

Epimorphism

A on to homomorphism is called epimorphism.

Isomorphism

An objective homomorphism is called isomorphism.

Endomorphism

A homomorphism mapping from $R \to R$ is called endomorphism.

Automorphism

A homomorphism $\phi: R \to R'$ is said to be automorphism if ϕ is bijective.

12. Kernel

Let $\phi: R \to R$` be a ring homomorphism, then kernel ϕ is defined as

$K\,\phi = \{ a \in R, \phi(a) = 0, 0 \in R \} \subseteq R$

13. Center of Ring

$C\,(R) = \{x \in R: xy = yx \text{ for all } y \in R\}$

14. Ideal

A non empty subset ***I*** of R said to be a two sided ideal.

(I) *I* is a sub group to R

(II) For $a \in I$ and $r \in \mathbf{R}$ both ar and ra in *I*.

Left ideal

A non empty subset *I* of R said to be left ideal in R if $ar \in I$.

Right ideal

A non empty subset *I* of R said to be right ideal in R if $ra \in I$.

Remarks:

A non empty subset commutative then every left ideal = right ideal and conversely

1. In general let ideal may not be right ideal and vice versa
2. Every ideal is a sub Ring.

Properties

Let ***I*** and ***J*** be two ideal of a ring **R** the

I. In ***J*** is an ideal of **R**.

II. ***I+J*** is an ideal of ***R.***

III. ***IJ*** is an ideal in **R.**

15. Quotient Ring

Let ***I*** be an ideal in **R.** For a∈**R** we define **R/*I*** as

R/*I* = { a+*I*, *a*∈R}

={ *I* + a, a∈R}

With addition and multiplication defined by

(a+*I*) + (b+*I*) = (a+b) + *I*

(a+*I*) (b+*I*) = ab + *I*

16. Maximal ideal

An ideal M≠R in a ring R said to be maximal ideal of R if there is an ideal U of R such that M⊆U⊆R

Then either M=U or U=R

17. Principal ideal

Let a≠0∈R then

I= Ra={ra; r∈R}

Is called principal ideal generated by a single element.

It is also called cyclic ideal or irreducible ideal.

Remark:

A ring is said to be a principal ideal ring au of its ideal are principal ideals.

18. Prime ideal

An ideal P≠R in a commutative ring R is said to be a prime ideal if a b∈P or b∈P

Remark:

If R is an integral Domain then {0} is a trivial prime ideal

19. First fundamental theorem of Ring Homomorphism

If φ:R→Sφ:R→S is a homomorphism of rings, then the kernel of φφ is an ideal of RR, the image of φφ is a subring of SS and R/kerφR/kerφ is isomorphic as a ring to φ(R)φ(R).

Proof: Let φ:R→Sφ:R→S be a ring homomorfism. If r∈Rr∈R and r′∈kerφr′∈kerφ, then we have rr′,r′r∈kerφrr′,r′r∈kerφ (so that it is closed under multiplication by elements of RR) since

φ(rr′)=φ(r)φ(r′)=φ(r)0=0=0φ(r)=φ(r′)φ(r)=φ(r′r);φ(rr′)=φ(r)φ(r′)=φ(r)0=0=0φ(r)=φ(r′)φ(r)=φ(r′r);

since kerφkerφ is also a subring of RR, it is an ideal of RR. It's clear that φ(R)φ(R) is a subring of SS. Now, let II be an ideal of RR, so that R/IR/I is also a ring, and define π:R→R/Iπ:R→R/I by π(r)=r+Iπ(r)=r+I. We know ππ is a group homomorphism with kernel II, and for r,s∈Rr,s∈R, we have

π(rs)=(rs)+I=(r+I) (s+I)=π(r)π(s),π(rs)=(rs)+I=(r+I) (s+I)=π(r)π(s),

so that ππ is in fact a ring homomorphism. Define then ϕ:R/kerφ→φ(R)ϕ:R/kerφ→φ(R) by

ϕ(r+(kerϕ))=φ(r),ϕ(r+(kerϕ))=φ(r),

for each (r+(kerϕ))∈R/kerφ(r+(kerϕ))∈R/kerφ, for some r∈Rr∈R. This is well defined because if r′∈(r+(kerφ)),r′∈(r+(kerφ)), then

ϕ(r′+(kerφ))=φ(r′)=φ(r)=ϕ(r+(kerφ)).ϕ(r′+(kerφ))=φ(r′)=φ(r)=ϕ(r+(kerφ)).

Also, this is a ring isomorphism because for each φ(s)∈φ(R)φ(s)∈φ(R) for some s∈Rs∈R, we have

(∗) ϕ−1{φ(s)}=ϕ−1φ[r+(kerφ)]=ϕ−1φ[π−1{r+(kerφ)}]={r+(kerφ)},
(∗) ϕ−1{φ(s)}=ϕ−1φ[r+(kerφ)]=ϕ−1φ[π−1{r+(kerφ)}]={r+(kerφ)},

a set with a single element of R/kerφR/kerφ, so that it is a bijection, and for every r+(kerφ),r′+(kerφ)∈R/kerφr+(kerφ),r′+(kerφ)∈R/kerφ, for some r,r′∈Rr,r′∈R, we have

ϕ[(r+(kerφ))+(r′+(kerφ))]=ϕ[(r+r′)+(kerφ)]=φ(r+r′ =φ(r)+φ(r′)=ϕ[r+(kerφ)]+ϕ[r′+(kerφ)],ϕ[(r+(kerφ))+(r′+(kerφ))]=ϕ[(r+r′)+(kerφ)]=φ(r+r′)=φ(r)+φ(r′)=ϕ[r+(kerφ)]+ϕ[r′+(kerφ)], and

$\phi[(r+(\ker\varphi))\ (r'+(\ker\varphi))]=\phi[(rr')+(\ker\varphi)]=\varphi(rr')=\varphi(r)\varphi(r')=\phi[r+(\ker\varphi)]\phi[r'+(\ker\varphi)],\phi[(r+(\ker\varphi))\ (r'+(\ker\varphi))]=\phi[(rr')+(\ker\varphi)]=\varphi(rr')=\varphi(r)\varphi(r')=\phi[r+(\ker\varphi)]\phi[r'+(\ker\varphi],$

so that it is a ring homomorphism.

20. Second fundamental theorem of Ring Homomorphism

Let GG be a group, let $H\leq G, H\leq G$, and let $N\trianglelefteq G. N\trianglelefteq G$. Then the set

$HN=\{hn:h\in H,n\in N\}HN=\{hn:h\in H,n\in N\}$

is a subgroup of G,G, $H\cap N\trianglelefteq H, H\cap N\trianglelefteq H$, *and*

$H/(H\cap N)\simeq HN/N$.

21. Third fundamental theorem of Ring Homomorphism

Let GG *be a group, and let* KK *and* NN *be normal subgroups of* G,G, *with* $K\subseteq N.K\subseteq N$. *Then* $N/K\trianglelefteq G/K, N/K\trianglelefteq G/K$, *and*

$(G/K)/(N/K)\simeq G/N$.

22. Applications of Fundamental Theorem of Ring Homomorphism

In abstract algebra, the fundamental theorem on Homomorphisms, also known as the fundamental Homomorphism theorem, relates the structure of two objects between which a Homomorphism is given, and of the kernel and image of the Homomorphism. The first isomorphism theorem follows from the category theoretical fact that the category of groups is (normal epi, mono) factorizable; in other words, the normal epimorphism and the monomorphism form a factorization system for the category. This is captured in the commutative diagram in the margin, which shows the objects and morphism whose existence can be deduced from the morphism f: G H. The diagram shows that every morphism in the category of groups has a kernel in the category theoretical sense; the arbitrary morphism f factors into i o π, where i is a monomorphism and π is an epimorphism (in a conormal category, all epimorphisms are normal). This is represented in the diagram by an object Kerf and a monomorphism k: Kerf G (kernels are always monomorphisms), which complete the short exact sequence running from the lower left to the upper of the diagram. The use of the exact sequence convention saves us from having to draw the zero morphism from kerj to H and kerj/ G.

If the sequence is right split (i.e. there is a morphism σ that maps G/Ker f to a π- pre-image of itself), then G is the semi direct product of the normal subgroup im k and the subgroup im σ. If it is left ssplit then it must also be right split, and im k× im σ is a direct product decomposition of G. In general, the existence of a right split does not imply the existence of a left split; but in an abelian category (such as the abelian groups), left splits and right splits are equivalent by the splitting lemma, and a right split is sufficient to produce a direct sum decomposition im k+ imσ. In an abelian category, all monomorphisms are also normal and the diagram may be extended by a second short exact sequence 0 G/ker f H coker f 0.

In the second isomorphism theorem, the product SN is the join of S and N in the lattice of subgroups of G, while the intersection S⋂N is the meet.

The third isomorphism theorem is generalized by the nine lemma to abelian categories and more general maps between objects. This isomorphism theorem has been called the " diamond theorem" due to the shape of the resulting subgroup lattice with SN at the top, S⋂N at the bottom and with N and S to the sides. It has even been called the " parallelogram rule" (by analog with the parallelogram rule for vectors) because in the resulting subgroup lattice the two sides assumed to represent the quotient groups (SN)/N and S/(S⋂N) are " equal" in the sense of isomorphism.

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3/16/2021