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Impulsive Response of Damped Harmonic Oscillator Via Fourier Transform

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Abstract: In this work, the impulsive response of the lightly damped harmonic oscillator was obtained by utilizing Fourier transform. The impulsive response was used to define the velocity, acceleration and mechanical energy of the lightly damped harmonic oscillator having Dirac delta function as damping force function. The theory of this lightly damped harmonic oscillator was extended to an overdamped harmonic oscillator with same damping force function (impulse function). The velocity, acceleration and mechanical energy for an overdamped oscillator has been also discussed. Fourier transform was used to reduce the repeated use of the initial boundary value conditions for ordinary differential equation consisting impulse function. Application of Fourier transform related to translated impluse function has been developed in this process.

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1. Introduction

A type of motion in which a body moves to and fro about a mean position is called oscillatory or vibratory motion. An important type of oscillatory motion is periodic motion or harmonic motion. The harmonic motion refers the repetition of a motion at regular intervals. In mechanics and physics, simple harmonic motion is a special type of periodic motion where the restoring force on the moving object is directly proportional to the object's displacement magnitude and acts towards the object's equilibrium position. It results in an oscillation which, if uninhibited by friction or any other dissipation of energy, continues indefinitely.

Simple harmonic motion can serve as a mathematical model for a variety of motions and can also be used to model molecular vibration as well. Simple harmonic motion provides a basis for the characterization of more complicated periodic motion through the techniques of Fourier analysis. Two well known types of harmonic motion are simple harmonic motion and damped harmonic motion that are defined by (Walker et al., 2011).

The simple harmonic motion refers the particular kind of harmonic motion where the periodic motion is cosine function. The cosine function is specific type of sinusoidal function, where sinusoidal function is the function of time. The displacement z of the object performing the simple harmonic motion from the origin is defined as a sinusoidal function by $z(t) = z_m cos(\omega t + \theta)$. Where for undamped harmonic oscillator z_m, ω, t and θ represents the amplitude, angular frequency, time and phase constant respectively.

Simple harmonic motion can be considered the one dimensional projection of uniform circular motion. If an object moves with angular speed ω around a circle of radius r centered at the origin of the xyplane, then its motion along each coordinate is simple harmonic motion with amplitude r and angular frequency ω. A mass m attached to a spring of spring constant k exhibits simple harmonic motion in closed space. The equation for describing the period $T = 2\pi\sqrt{\frac{m}{k}}$, shows the period of oscillation is independent of the amplitude, though in practice the amplitude should be small. This equation is also valid in the case when an additional constant force is being applied on the mass, i.e. the additional constant force cannot change the period of oscillation.

In the small-angle approximation, the motion of a simple pendulum is approximated by simple harmonic motion. The period of a mass attached to a pendulum of length l with gravitational acceleration g is given by $T = 2\pi\sqrt{\frac{l}{g}}$. This shows that the period of oscillation is independent of the amplitude and mass of the pendulum but not of the acceleration due to gravity, g, therefore a pendulum of the same length on the Moon would swing more slowly due to the Moon's lower gravitational field strength. Because the value of g varies slightly over the surface of the earth, the time period will vary slightly from place to place and will also vary with height above sea level (Triana et al., 2011).

The presence of an external force reduces the motion of the oscillator, this external force is known as damping force. The motion produce by damping force refers the damped simple harmonic motion. The further cases for damped oscillation are under damping, over damping and critical damping. The condition in which damping of an oscillator causes it to return to equilibrium with the amplitude gradually decreasing to zero; system returns to equilibrium faster but overshoots and crosses the equilibrium position one or more times is associated with under damped system. The condition in which damping of an oscillator causes it to return to equilibrium without oscillating; oscillator moves more slowly toward equilibrium than in the critically damped system is associated with over damped system. Critical damping is defined as the threshold between over damping and underdamping. In the case of critical damping, the oscillator returns to the equilibrium position as quickly as possible, without oscillating, and passes it once at most.

The displacement z of the damped oscillator performing damped simple harmonic motion is defined as $z(t) = z_m e^{\frac{-\beta t}{2m}} \cos(\omega_0 t + \theta)$. Where for damped harmonic oscillator z_m, β, t, m, $\omega_o,\ t\ and\ \theta$ amplitude, damping constant, time, mass, angular frequency and phase constant respectively. When the damped oscillator is further affected due to the presence of an external force $f(t)$ then the damped oscillator changes into a driven harmonic oscillator. When the parameters of oscillator that are restoring force or damping are varied and driven energy is given by these parameters then the oscillator is called as a parametric oscillator. Basically, a driven harmonic oscillator gives further motivation for the parametric oscillator. The classical varactor parametric oscillator and the Optical parametric oscillator are the examples of the parametric oscillator (Poonyawatpornkul et al., 2013).

The action of a single force z on the mass m of an oscillator gives the concept of simple harmonic oscillator and makes it different from damped and from driven harmonic oscillators. The mass is pulled towards the point $z = 0$ by the force. The dependence of force involves only two terms that are position of mass (say) z and a constant C. The mass always displaces in the opposite direction of the exerted force, therefore for this reason displacement and force are always taken in opposite direction.

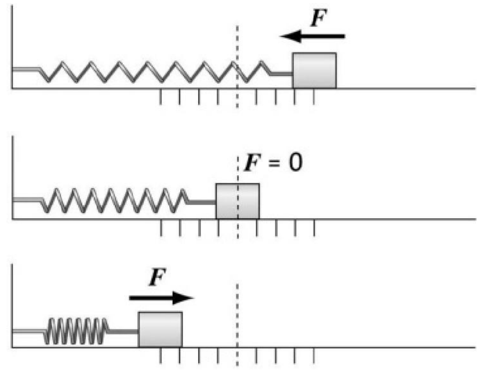

Figure 1: A simple harmonic oscillator.

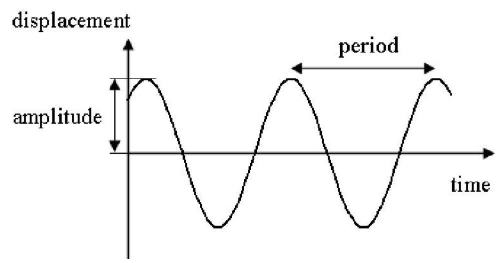

Figure 2: Period, displacement and amplitude of simple harmonic oscillator.

The expression for this force is $z = -C_z$ by equalizing this force with Newton's second law $z = ma$ the differential equation for simple harmonic oscillator is found $ma = -C_z$ or $ma + C_z = 0$ or $a + \frac{C}{m} z = 0$ with value of acceleration $a = \frac{d^2 z}{dt^2}$ the final form of differential equation comes as:

$$\frac{d^2 z}{dt^2} + \frac{C}{m} z = 0 \qquad (1)$$

The solutions for the above differential equation are obtained that are,

$$z = z_m cos(\omega t + \theta) \text{ Or } z = z_m \sin(\omega t + \theta)$$

Where $\omega = \frac{C}{m}$, t represents time and z_m is change in displacement due to application of force and known as amplitude and θ represents phase constant. The both solutions are periodic functions with their respective sinusoidal functions and are referred as the responses displacements at specific time t varying quantity $\omega t+\theta$ of simple harmonic oscillator. The amplitude z_m always maintains its value as maximum due to absence of any external dissipate forcing function that may cause damping.

The harmonic oscillator model is very important in physics, because any mass subject to a force in stable equilibrium acts as a harmonic oscillator for small vibrations. Harmonic oscillators occur widely in nature and are exploited in many man made devices, such as clocks and radio circuits. They are the source of virtually all sinusoidal vibrations and waves. Mechanical examples include pendulums (with small angles of displacement), masses connected to springs, and acoustical systems. Other analogous systems include the electrical harmonic oscillators such as RLC circuits (Rushka et al., 2019).

The displacement for equation (1) of simple harmonic oscillator is defined in the following mannerism that in fact solutions which have already been obtained by solving equation of simple harmonic oscillator (Rushka et al., 2019),

$$d = z_m cos\left(\omega t+\theta\right) \text{ Or } d = z_m \sin\left(\omega t+\theta\right)$$

For displacement $d = z_m cos\left(\omega t+\theta\right)$ the expression for velocity of simple harmonic oscillator is found by differentiating it with respect to time function t, that has following periodic expression,

$$V = \frac{dz}{dt} = -\omega z_m \sin\left(\omega t+\theta\right)$$

By taking $\omega z_m = v_m$ that represents the amplitude for velocity of the harmonic oscillator and treated same as z_m the amplitude of harmonic oscillator motion it obtains that,

$$V = -v_m \sin\left(\omega t+\theta\right)$$

the velocity changes between the limits $-\omega z_m$ and $+\omega z_m$ and at maximum value of the displacement d the velocity v attains its minimum value and similarly at maximum value of the velocity v the displacement d attains its minimum value. The reverse order relation of the velocity with the displacement of the simple harmonic oscillator is figured in (3) studied from (Rushka et al., 2019).

For the velocity $V = -\omega z_m \sin\left(\omega t+\theta\right)$, the expression for acceleration of simple harmonic oscillator is found by again repeating the same previous process of differentiation with respect to time function t, that is an oscillatory function,

$$a = \frac{dv}{dt} = -\omega^2 z_m \cos\left(\omega t+\theta\right) = -\omega^2 d$$

The above hallmark relation $a = -\omega^2 d$ of the simple harmonic oscillator varies between the values of its amplitude that points towards non-negative term $a = -\omega^2 z$ that is considered as am. The acceleration a is proportional to the displacement d with changing sign of d (Rushka et al., 2019).

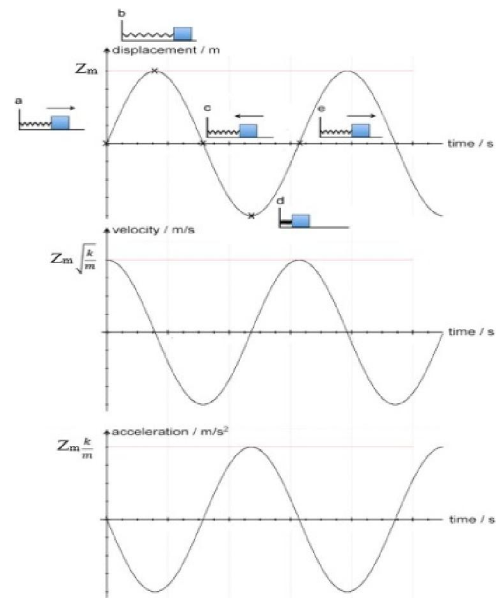

Figure 3: Displacement, velocity and acceleration of simple har- monic oscillator.

Equation (1) in terms of velocity v of simple harmonic oscillator is solved by utilizing its another form in the following way,

$$\frac{d}{dt}\left(\frac{dz}{dt}\right) + \frac{C}{m}z = 0 \ , \ \frac{d}{dt}V + \frac{C}{m}z = 0$$

$$\frac{d}{dt}V = -\frac{C}{m}z$$

Since $V = \frac{dz}{dt}$, and so multiplying above equation with $Vdt = dz$ it is obtained that,

$$Vdt \times \frac{d}{dt}V = -\frac{C}{m}z \ , \ mVdV = -Czdz$$

Since $d\left(z^2\right) = 2zdz$ and $d\left(V^2\right) = 2VdV$ then above equation becomes,

$$d\left(\frac{1}{2}mV^2\right)=-d\left(\frac{1}{2}Cz^2\right)$$

by integrating above equation it becomes,

$$\left(\frac{1}{2}mV^2\right)+\left(\frac{1}{2}Cz^2\right)=k$$

Where k represents the constant of integration, the kinetic energy K.E of the mass of simple harmonic oscillator is due to its motion while potential energy P.E of the mass comes from the spring capacity for being stretched or compressed. In extending or compressing U = P.E = z.d, extending from z to z + dz implies d = dz so that P.E = (Cz)dz. Hence work done that is P.E when mass m is extended from its unstretched length by an amount z then,

$$P.E=\int_0^z Czdz=\frac{1}{2}Cz^2$$

Now by utilizing values of $K.E=\frac{1}{2}mV^2$ and $P.E=\frac{1}{2}Cz^2$ in the expression of total energy T.E = K.E + P.E it obtains that,

$$T.E=\frac{1}{2}Cz_m^2$$

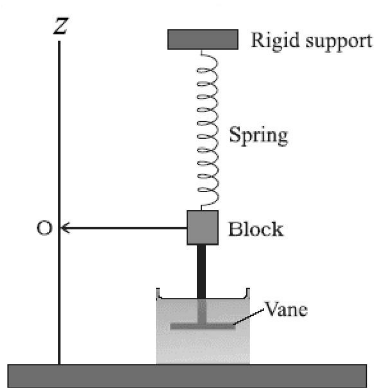

Figure 4: Energy of simple harmonic oscillator.

At any position the total energy $T.E$ of the simple harmonic oscillator is constant, this energy feature of the motion of simple harmonic oscillator has great importance due to scalar consideration that solves physical problems by simplifying the analysis (Rushka et al., 2019). The presence of an external force F' causes the reduction of mechanical energy of an oscillator that further reduces the amplitude of oscillations. This external agency is commonly the friction force and refers in general as the damping force function. Damping forcing function F' plays its role as dissipate force and proportional to velocity $\vec{v}$, the extensions are made in the simple harmonic oscillatory system by introducing a vane and a liquid along with already introduced block mass m and spring with C coefficient of constant. The relation between velocity and dissipate force is $F'\propto\vec{v}$ that becomes $F'\propto l_0\vec{v}$, after utilization the conception about liquid and vane the opposition of dissipate force F' for the oscillatory motion is described by negative sign that comes with coefficient of damping constant lo that is a combined property of both liquid and vane.

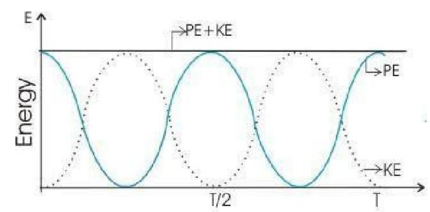

Figure 5: Simple damped harmonic oscillator.

The combined action of forces on mass of oscillator is equal to $-l_0\vec{v}$ and $-Cz$ which points towards the dissipate force F' and spring force f_s respectively. The utilization of Newton's second law of motion $ma_z=F_z$ along the z component of motion for this system produces the following results,

$$ma=-l_0\vec{v}-Cz$$

Putting original expressions for acceleration a and velocity v that are associated with motion of oscillator along $z-axis$ direction, it founds the differential equation for simple damped harmonic oscillator changes as:

$$m\frac{d^2z}{dt^2}+l_0\frac{dz}{dt}+Cz=0$$

the simplification of the above differential equation produces the solution of simple damped harmonic oscillator that is,

$$z(t)=z_me^{\frac{-\beta t}{2m}}\cos(\omega_0t+\theta)$$

Angular frequency and amplitude of simple damped harmonic oscillator are represented by ω_0 and $z_me^{\frac{-\beta t}{2m}}$ respectively, the value of angular frequency involves and depends on the following terms,

$$\omega_0 = \sqrt{\frac{C}{m} - \frac{l_o^2}{4m^2}}$$

The condition of undamped harmonic oscillator holds in the present situation if the damping coefficient of constant reaches to its zero value that is $l_o = 0$ then $\omega_o = \sqrt{\frac{C}{m}}$ and displacement of simple damped harmonic oscillator reduces into following expression,

$$z(t) = z_m cos(\omega t + \theta)$$

that is specific for undamped oscillator for small dissipate forcing function $l_o << \sqrt{\frac{C}{m}}$ m the damping phenomenon has so small effects on oscillator therefore, the term $z_m e^{\frac{-\beta t}{2m}}$ is considerable as z_m that is being decreasing with time and so it results into $\omega_o \approx \omega$. For small damping the energy of harmonic oscillator does not remain same for both cases that are damped and undamped rather for damped harmonic oscillator it is decreasing as basically the reduction in amplitude happens with the increment in time variable therefore, total energy described in (Kharkongor et al., 2018),

$$E(t) \approx 0.5cz_m^2 e^{\frac{-\beta t}{2m}}$$

Theoretical Consideration

The nature of the damped harmonic oscillator depends upon the nature of damping force. The damping force may be taken as any type of the periodic force. Impulse function is an example of the damping force, for continuous-time systems this impulse function is a Dirac delta function $\delta(t)$ while for the discrete-time systems it is the Kronecker delta (Chen, 2014). A damped harmonic oscillator for which the damping force is an impulsive force is called the lightly damped harmonic oscillator. The product of Dirac delta function $\delta(t)$ and its strength η gives this impulsive force. The lightly damped harmonic oscillator for which the forcing function is an impulsive force has the following differential equation represented as:

$$z^n(t) + 2\alpha z'(t) + \omega_o^2 z(t) = F(t)$$

Where the impulsive force $z(t)$ is the product of a Dirac delta function $\delta(t)$ and the strength of the Dirac delta function η. The dimensions of η are identical as that of velocity that is $F(t) = \eta\delta(t)$ represents the change in momentum per unit mass (Gupta et al., 2019).

The Laplace transform of the function $z(t)$ with n^{th} derivative is given as following in (Ebaid et al., 2015),

$$L\{z^n(t)\} = s^n Z(s) - s^{n-1}z(0) - s^{n-2}z'(0) - s^{n-3}z''(0) - ... - z^{n-1}(0)$$

Where $L\{z^n(t)\} = Z(s) = \int_0^\infty e^{-st} z(t)dt;$

$t \geq 0$, The convolution for two functions, $\upsilon(\tau)$ and $\psi(\tau)$ denoted by, $(\upsilon * \psi)(\tau)$ and is defined as:

$$(\upsilon * \psi)(\tau) = \int_0^\tau \upsilon(t)\psi(\tau - t)dt, \quad \tau \geq 0 \qquad (2)$$

Where the both functions υ(τ) and ψ(τ) are piecewise continuous in the interval of [o, ∞[.

The Laplace transform of equation (2) provided that the functions, υ(τ) and ψ(τ) are exponential order, is represented in (Widder, 2015),

$$L\{(\upsilon*\psi)(\tau)\} = L\{\upsilon(\tau)\}L\{\psi(\tau)\} = \bar{\upsilon}(\tau)\bar{\psi}(\tau)$$

Where $\bar{\upsilon}(\tau)$ and $\bar{\psi}(\tau)$ represent the Laplace transforms of, υ(τ) and ψ(τ) respectively, also the Laplace transform operator is denoted here by L. Therefore, the inverse Laplace transform of [$\bar{\upsilon}(\tau)$ $\bar{\psi}(\tau)$] is,

$$L^{-1}\left[\bar{\upsilon}(\tau)\bar{\psi}(\tau)\right] = \left[(\upsilon * \psi)(\tau)\right].$$

The utilization of the above theorem helps in the calculation of impulsive response for the lightly damped harmonic oscillator. The displacement for the lightly damped harmonic oscillator is determined by convolution theory of Laplace transform along with the utilization of the Laplace transform of the function z (t) with n^{th} derivative theorem as:

$$z(t) = \frac{\nu_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sin\sqrt{\omega_o^2 - \alpha^2 t}$$

For $\alpha > \omega_o$ the damped harmonic oscillator is subjected to overdamped harmonic oscillator. The impulsive response denoted as $z'(t)$ of overdamped oscillator is also determined successfully by similar algorithm of Laplace transformation that already has been used for lightly damped harmonic oscillator, therefore the response of the overdamped harmonic oscillator is obtained as given (Gupta et al., 2019),

$$z'(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\alpha^2 - \omega_o^2}} \sinh\sqrt{\alpha^2 - \omega_o^2 t}$$

Here the impulsive response $z(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sin\sqrt{\omega_o^2 - \alpha^2 t}$ of lightly damped harmonic oscillator is again discussed by using Fourier transform. The velocity and the acceleration for the impulsive response are evaluated. The expression for mechanical total energy T.E of the lightly damped harmonic oscillator is also obtained. Fourier transform is further applied on an overdamped harmonic oscillator and same solution $z'(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\alpha^2 - \omega_o^2}} \sinh\sqrt{\alpha^2 - \omega_o^2 t}$ is found with simple computational layout. Translated lightly damped harmonic oscillator is introduced and studied by Fourier transform with its differential equation,

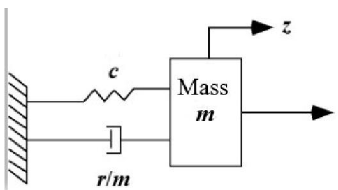

Figure 6: Lightly damped harmonic os- cillator.

$$z''(t) + 2\alpha z'(t) + \omega_o^2 z(t) = \eta\delta_b(t).$$

The damped harmonic oscillator for which the forcing function is an impulsive force has the following differential equation as:

$$z''(t) + 2\alpha z'(t) + \omega_o^2 z(t) = F(t)$$

Where the impulsive force z (t) is the product of a Dirac delta function δ (t) and the strength of the Dirac delta function η. The dimensions of η are identical as that of velocity that is,

$$F(t) = \eta\delta_b(t)$$

Represents the change in momentum per unit mass. The Dirac delta function δ(t) is subjected to impulse and the forcing function (impulsive force) is performing for a very short period of time. The damping constant per unit mass r m is represented by 2α and has dimensions similar to frequency. The natural frequency ω_o of the damped harmonic oscillator is represented by $\sqrt{\frac{C}{m}}$ .If the mechanical vibration α of the damped harmonic oscillator is less than its natural frequency $\alpha < \omega_o$ then the oscillator refers to the lightly damped harmonic oscillator. For the calculation of the impulsive response z (t) of a lightly damped harmonic oscillator, the initial boundary value conditions are:

Results and Discussion

Consider the ordinary differential equation for the damped harmonic oscillator,

$$z''(t) + 2\alpha z'(t) + \omega_o^2 z(t) = F(t) \quad (3)$$

Since forcing function is taken as $F(t) = \eta\delta_b(t)$, so equation (3) can be written in the form of impulsive force function as:

$$z''(t) + 2\alpha z'(t) + \omega_o^2 z(t) = \eta\delta(t) \quad (4)$$

By taking Fourier transform on both sides of equation (4) it obtains,

$$F\{z''(t) + 2\alpha z'(t) + \omega_o^2 z(t)\} = F\{\eta\delta(t)\}$$

$$F\{z''(t)\} + 2\alpha F\{z'(t)\} + \omega_o^2 F\{z(t)\} = \eta F\{\delta(t)\} \quad (5)$$

As the Fourier transform of a continuous function z (x) on x-axis with k^{th} derivative is,

$$F\{z^k(x)\} = (iw)^k \bar{z}(w), \quad k = 1, 2, 3, ???$$

Where,

$$F\{z(x)\} = \bar{z}(w) = \int_{-\infty}^{\infty} e^{-iwx} z(x)\, dx;$$

$$-\infty < w < \infty$$

Moreover, the function $z^k(x)$ is absolutely integrable on $-\infty < w < \infty$ and piecewise continuous. Also the Fourier transform of Dirac delta function provides, $F\{\delta(t)\} = 1$. By applying Fourier transform theory on the functions $z''(t), z'(t)$ and $z(t)$ it is found the following results,

$$F\{z''(t)\} = (iw)^2 \bar{z}(w)$$
$$F\{z'(t)\} = (iw)\bar{z}(w)$$

- When the damped harmonic oscillator is passing through its mean position then the time t is measured from this instant and at the time $t=0$ for which the impulsive response becomes zero $z(0)=0$.
- For an instant $t=0^+$ which is the time just after the application of the impulsive force $z(t)$, the velocity of the damped harmonic oscillator becomes maximum and assumed to be $z'(0^+)=v_o$ z 0 (0+) = vo (Gupta et al., 2019).

$$F\{z(t)\} = \bar{z}(w)$$

now by using the above results the equation (5) gives yield as:

$$(iw)^2 \bar{z}(w) + 2\alpha(iw)\bar{z}(w) + \omega_o^2 \bar{z}(w) = \eta(1)$$
$$\left\{(iw)^2 + 2\alpha(iw) + \omega_o^2\right\}\bar{z}(w) = \eta(1) \qquad (6)$$

$$(iw+\alpha)^2 - \left(i\sqrt{\omega_o^2-\alpha^2}\right)^2 = (iw)^2 + \alpha^2 + 2\alpha iw - i^2\left(\omega_o^2-\alpha^2\right)$$
$$= (w)^2 + \alpha^2 + 2\alpha w + \left(\omega_o^2 - \alpha^2\right)$$
$$= (iw)^2 + 2\alpha iw + \omega_o^2 - \alpha^2 + \alpha^2$$
$$= (iw)^2 + 2\alpha iw + \omega_o^2 + 0$$
$$= (iw)^2 + 2\alpha iw + \omega_o^2 \qquad (7)$$

using result of the above consideration (7) in equation (6),

$$\left\{(iw+\alpha)^2 - \left(i\sqrt{\omega_o^2-\alpha^2}\right)^2\right\}\bar{z}(w) = \eta$$

$$\left\{(iw+\alpha) - \left(i\sqrt{\omega_o^2-\alpha^2}\right)\right\}\left\{(iw+\alpha) + \left(i\sqrt{\omega_o^2-\alpha^2}\right)\right\}\bar{z}(w) = \eta \qquad (8)$$

Let $\lambda_1 = \alpha + i\sqrt{\omega_o^2-\alpha^2}$ (9)

And let $\lambda_2 = \alpha - i\sqrt{\omega_o^2-\alpha^2}$ (10)

$$\lambda_1 - \lambda_2 = \alpha + i\sqrt{\omega_o^2-\alpha^2} - \left\{\alpha - i\sqrt{\omega_o^2-\alpha^2}\right\}$$
$$\lambda_1 - \lambda_2 = \alpha + i\sqrt{\omega_o^2-\alpha^2} - \alpha + i\sqrt{\omega_o^2-\alpha^2}$$
$$= \alpha - \alpha + 2i\sqrt{\omega_o^2-\alpha^2}$$
$$= 0 + 2i\sqrt{\omega_o^2-\alpha^2}$$
$$\lambda_1 - \lambda_2 = 2i\sqrt{\omega_o^2-\alpha^2} \qquad (11)$$

After substituting equations (9) and (10) in equation (8) it finds that,

$$(iw+\gamma_1)(iw+\gamma_2)\bar{z}(w) = \eta$$
$$\bar{z}(w) = \frac{\eta}{(iw+\gamma_1)(iw+\gamma_2)}$$

$$\bar{z}(w) = \eta\left[\frac{1}{(iw+\gamma_1)(iw+\gamma_2)}\right] \qquad (12)$$

consider the following term,

$$\frac{1}{\gamma_2-\gamma_1}\left\{\frac{1}{iw+\gamma_1} - \frac{1}{iw+\gamma_2}\right\} = \frac{1}{\gamma_2-\gamma_1}\left\{\frac{iw+\gamma_2-(iw+\gamma_1)}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$
$$= \frac{1}{\gamma_2-\gamma_1}\left\{\frac{iw+\gamma_2-iw-\gamma_1}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$
$$= \frac{1}{\gamma_2-\gamma_1}\left\{\frac{iw-iw+\gamma_2-\gamma_1}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$
$$= \frac{1}{\gamma_2-\gamma_1}\left\{\frac{0+\gamma_2-\gamma_1}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$
$$= \frac{\gamma_2-\gamma_1}{\gamma_2-\gamma_1}\left\{\frac{1}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$
$$= 1.\left\{\frac{1}{(iw+\gamma_1)(iw+\gamma_1)}\right\}$$

So, $$\frac{1}{\gamma_2-\gamma_1}\left\{\frac{1}{iw+\gamma_1} - \frac{1}{iw+\gamma_2}\right\} = \frac{1}{(iw+\gamma_1)(iw+\gamma_1)} \qquad (13)$$

by substituting equation (13) in equation (12) it obtains,

$$\bar{z}(w) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{\frac{1}{iw+\gamma_1} - \frac{1}{iw+\gamma_2}\right\}$$

By taking inverse Fourier transform of above expression,

$$F^{-1}\{\bar{z}(w)\} = \frac{\eta}{\gamma_2 - \gamma_1} F^{-1}\left\{\frac{1}{iw+\gamma_1} - \frac{1}{iw+\gamma_2}\right\}$$

$$z(w) = \frac{\eta}{\gamma_2 - \gamma_1}\left[F^{-1}\left\{\frac{1}{iw+\gamma_1}\right\} - F^{-1}\left\{\frac{1}{iw+\gamma_1}\right\}\right] \quad (14)$$

Since $F^{-1}\left\{\frac{1}{iw+\gamma_1}\right\} = \begin{cases} 0 & if \quad t<0 \quad (\lambda>0) \\ e^{-\lambda t} & if \quad t>0 \end{cases}$

hence the equation (14) implies,

$$z(w) = \frac{\eta}{\gamma_2 - \gamma_1}\begin{cases} 0 & if \quad t<0 \quad (\lambda_1>0) \\ e^{-\lambda t} & if \quad t>0 \end{cases} - \frac{\eta}{\gamma_2 - \gamma_1}\begin{cases} 0 & if \quad t<0 \quad (\lambda_2>0) \\ e^{-\lambda t} & if \quad t>0 \end{cases}$$

Since the impulsive response of lightly damped harmonic oscillator is examined for t > 0 so function z (t) attains the values for t > 0,

$$z(t) = \frac{\eta}{\gamma_2 - \gamma_1} e^{-\gamma_1 t} - \frac{\eta}{\gamma_2 - \gamma_1} e^{-\gamma_2 t}$$

$$z(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{e^{-\gamma_1 t} - e^{-\gamma_2 t}\right\} \quad (15)$$

By differentiating equation (15) with respect to time t it becomes,

$$\frac{d}{dt} z(t) = \frac{\eta}{\gamma_2 - \gamma_1} \frac{d}{dt}\left\{e^{-\gamma_1 t} - e^{-\gamma_2 t}\right\}$$

$$z'(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{\frac{d}{dt} e^{-\gamma_1 t} - \frac{d}{dt} e^{-\gamma_2 t}\right\}$$

$$z'(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{-\gamma_1 e^{-\gamma_1 t} + \gamma_2 e^{-\gamma_2 t}\right\}$$

For the time t = 0 the above equation becomes, dc

$$z'(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{-\gamma_1 e^{-\gamma_1 0} + \gamma_2 e^{-\gamma_2 0}\right\}$$

$$z'(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{-\gamma_1 .1 + \gamma_2 .1\right\}$$

$$z'(t) = \frac{\eta}{\gamma_2 - \gamma_1}\left\{-\gamma_1 + \gamma_2\right\}$$

$$z'(t) = \eta$$

As for time $t = 0,\ z'(0) = v_o$ so from above expression it obtains that,

$$v_o = \eta \quad (16)$$

by using equation (16) in equation (15) it finds,

$$z(t) = \frac{v_o}{\gamma_2 - \gamma_1}\left\{e^{-\gamma_1 t} - e^{-\gamma_2 t}\right\} \quad (17)$$

By using suppositions (9), (10) and (11) in equation (17) it changes into the following form,

$$z(t) = \frac{v_o}{-2i\sqrt{\omega_o^2 - \alpha^2}}\left\{e^{-(\alpha + i\sqrt{\omega_o^2 - \alpha^2})t} - e^{-(\alpha - i\sqrt{\omega_o^2 - \alpha^2})t}\right\}$$

$$= \frac{v_o}{-2i\sqrt{\omega_o^2 - \alpha^2}}\left\{e^{-\alpha t} e^{-i\sqrt{\omega_o^2 - \alpha^2}t} - e^{-\alpha t} e^{i\sqrt{\omega_o^2 - \alpha^2}t}\right\}$$

$$= \frac{v_o e^{-\alpha t}}{-2i\sqrt{\omega_o^2 - \alpha^2}}\left\{e^{-i\sqrt{\omega_o^2 - \alpha^2}t} - e^{i\sqrt{\omega_o^2 - \alpha^2}t}\right\}$$

$$= \frac{v_o e^{-\alpha t}}{2i\sqrt{\omega_o^2 - \alpha^2}}\left\{e^{i\sqrt{\omega_o^2 - \alpha^2}t} - e^{-i\sqrt{\omega_o^2 - \alpha^2}t}\right\}$$

$$z(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}}\left\{\frac{e^{i\sqrt{\omega_o^2 - \alpha^2}t} - e^{-i\sqrt{\omega_o^2 - \alpha^2}t}}{2i}\right\} \quad (18)$$

in exponential form the sin function has representation as:

$$\sin\sqrt{\omega_o^2 - \alpha^2}\,t = \frac{e^{i\sqrt{\omega_o^2 - \alpha^2}t} - e^{-i\sqrt{\omega_o^2 - \alpha^2}t}}{2i} \quad (19)$$

By putting equation (19) in equation (18) the following result is found,

$$z(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sin\sqrt{\omega_o^2 - \alpha^2}\,t \quad (20)$$

The above equation (20) shows an impulsive response of a lightly damped harmonic oscillator and describes that the impulsive response of the lightly damped harmonic oscillator is free from the strength of impulsive force $F(t)$. Also the oscillation has the amplitude that decreases with time exponentially and not constant. Thus, the motion of the lightly damped harmonic oscillator does not remain constant and slow down with time exponentially. Impulsive response z (t) for the lightly damped harmonic oscillator is the periodic function it is equal to the displacement of the lightly damped harmonic oscillator. Hence equation (20) refers the displacement $\vec{d}(t)$ which is equal to impulsive response z (t),

$$\vec{d}(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sin\sqrt{\omega_o^2 - \alpha^2 t} \quad (21)$$

Where the displacement function $\vec{d}(t)$ is sinusoidal function in sine function and the expression $\frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}}$ represents amplitude A of this oscillatory response $\vec{d}(t)$, therefore the displacement of lightly damped harmonic oscillator can be expressed as:

$$\vec{d}(t) = A\sin\sqrt{\omega_o^2 - \alpha^2 t} \quad (22)$$

$$\vec{v}(t) = -\alpha\vec{d}(t) + v_o e^{-\alpha t}\cos\sqrt{\omega_o^2 - \alpha^2 t} \quad (23)$$

The above equation (23) shows the velocity for the lightly damped harmonic oscillator. Moreover, it reveals that the velocity the lightly damped harmonic oscillator is the combination of two different sinusoidal functions those are the cosine and sine.

$$\vec{a}(t) = (2\alpha^2 - \omega_o^2)\vec{d}(t) - 2\alpha v_o e^{-\alpha t}\cos\sqrt{\omega_o^2 - \alpha^2 t} \quad (24)$$

The above equation (24) express the acceleration for the lightly damped harmonic oscillator and shows that it is sinusoidal function of the combination of cosine and sine functions. The acceleration of the lightly damped harmonic oscillator is oscillatory function.

By utilizing the conception about the total mechanical energy of the simple damped harmonic oscillator that implies half of the product of square of amplitude $z_m e^{\frac{-\beta t}{2m}}$ and coefficient of spring constant C refers total energy $E(t) \approx \frac{1}{2} C z_m e^{\frac{-\beta t}{2m}}$ for response $z(t) = z_m e^{\frac{-\beta t}{2m}} cos(\omega_o t + \theta)$, the total mechanical energy of the lightly damped harmonic oscillator can be evaluated by considering and substituting following terms in the expression $T.E = K.E + P.E = \frac{1}{2}m\vec{v}^2 + \frac{1}{2}Cz^2(t)$, therefore here by putting value of $\vec{v}$ from equation (23) in the expression of K.E it founds,

$$K.E = \frac{1}{2}m\left[\alpha^2\vec{d}^2(t) + v_o^2 e^{-2\alpha t}\cos^2\sqrt{\omega_o^2 - \alpha^2 t} - 2\alpha\vec{d}(t)v_o e^{-\alpha t}\cos\sqrt{\omega_o^2 - \alpha^2 t}\right] \quad (25)$$

$$P.E = \frac{1}{2}m\omega_o^2\vec{d}^2(t) \quad (26)$$

The potential energy as well as kinetic energy cause the oscillations for the system while the damping forcing function z minimizes these oscillation to its least value,

Substitution of equations (25) and (26) in the total energy expression T.E = P.E + K.E gives that,

$$T.E = \frac{1}{2}m\left[\left(\omega_o^2 - \alpha^2\right)\frac{v_o^2 e^{-2\alpha t}}{\omega_o^2 - \alpha^2}\sin^2\sqrt{\omega_o^2 - \alpha^2 t}\right] + \frac{1}{2}m\left[v_o^2 e^{-2\alpha t}\cos^2\sqrt{\omega_o^2 - \alpha^2 t}\right]$$
$$+\frac{1}{2}m\left[-2\alpha\frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}}\sin\sqrt{\omega_o^2 - \alpha^2 t} \times v_o e^{-\alpha t}\cos\sqrt{\omega_o^2 - \alpha^2 t}\right] \quad (27)$$

suppose that $\omega_o^2 + \alpha^2 = \omega'$ and $\omega_o^2 - \alpha^2 = \omega''$ that have their original forms $\frac{C}{m} + \frac{r^2}{4m^2} = \omega'$ and $\frac{C}{m} - \frac{r^2}{4m^2} = \omega''$ respectively, since this oscillator is subjected for $\alpha < \omega_o$ that implies for small damping natural frequency conserves because value of damping constant is too small as compare to the value of frequency $r << \sqrt{\frac{C}{m}}$, therefore it can be neglected and it founds that $\frac{C}{m} \approx \omega'$ and $\frac{C}{m} \approx \omega''$ finally it becomes $\omega_o^2 \approx \omega'$ and $\omega_o^2 \approx \omega''$ so heuristic use of transitive property of approximately for numbers yields $\omega' \approx \omega_o^2 \approx \omega''$ thus the above equation (27) changes into,

$$T.E \approx \frac{1}{2}CA^2 \quad (28)$$

Where $A = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}}$ is the amplitude of the lightly damped harmonic oscillator for response $\vec{d}(t)$. The above equation (28) describes that total mechanical energy does not conserve rather it reduces with the increment in time variable and it depends upon amplitude, also similar to amplitude it has scalar nature.

For $\alpha > \omega_o$ the damped harmonic oscillator is subjected to the overdamped harmonic oscillator. The impulsive response denoted as $z'(t)$ of the overdamped oscillator is determined successfully by replacing the term $\sqrt{\omega_o^2 - \alpha^2}$ of the equation (20) with $i\sqrt{\omega_o^2 - \alpha^2}$, so the equation (20) implies,

$$z'(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sinh \sqrt{\omega_o^2 - \alpha^2 t} \quad (29)$$

The equation (29) shows the impulsive response for the overdamped harmonic oscillator. It is also observed that the overdamped harmonic oscillator performs a periodic motion, because it contains sinh function in its expression that is not an oscillatory function.

Impulsive response $z'(t)$ for the Overdamped harmonic oscillator is an non-oscillatory function and it is equal to the displacement of the Overdamped harmonic oscillator. Hence equation (29) refers the displacement $\vec{d}'(t)$ which is identical to impulsive response $z'(t)$.

$$\vec{d}'(t) = \frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}} \sinh \sqrt{\omega_o^2 - \alpha^2} \quad (30)$$

Above equation (30) shows that displacement $\vec{d}'(t)$ is aperiodic function and the expression $\frac{v_o e^{-\alpha t}}{\sqrt{\omega_o^2 - \alpha^2}}$ represents amplitude A' of this aperiodic response $\vec{d}'(t)$, therefore it can be expressed as:

$$\vec{d}'(t) = A' \sinh \sqrt{\omega_o^2 - \alpha^2} \quad (31)$$

$$v'(t) = -\alpha \vec{d}'(t) + v_o e^{-\alpha t} \cosh \sqrt{\alpha^2 - \omega_o^2 t} \quad (32)$$

The above equation (32) shows the velocity for the overdamped harmonic oscillator is the combination of two different aperiodic function. Moreover, it reveals that similar to displacement $\vec{d}'(t)$ of the overdamped harmonic oscillator the velocity $v'(t)$ is also a nonoscillatory function.

$$\vec{a}'(t) = (2\alpha^2 - \omega_o^2)\vec{d}'(t) - 2\alpha v_o e^{-\alpha t} \cosh \sqrt{\alpha^2 - \omega_o^2 t} \quad (33)$$

The above equation (33) shows the acceleration for the overdamped harmonic oscillator and reveals that it is also non-oscillatory function. The acceleration is also the combination of two different non-oscillatory functions.

$$T.E < \frac{1}{2}CA'^2 \left[\cosh 2\sqrt{\omega_o^2 + \alpha^2 t} \quad \frac{2\alpha}{\sqrt{\omega_o^2 + \alpha^2}} \sinh 2\sqrt{\alpha^2 - \omega_o^2 \alpha t} \right] \quad (34)$$

The above equation (34) reveals the reduction phenomenon of energy for aperiodic overdamped system that happened due to $\alpha > \omega_o$. Total energy for this type of oscillator decreases by product of amplitude A' and aperiodic functions that is further the combination of two different non-oscillatory functions.

Reexamine the above discussed damped harmonic oscillator with translated forcing function, so that, impulsive force has changed into the shifted Dirac delta function from origin to a point (say) b on the time axis, the following expression represents is its differential equation,

$$z''(t) + 2\alpha z'(t) + \omega_o^2 z(t) = \eta\delta_b(t) \quad (36)$$

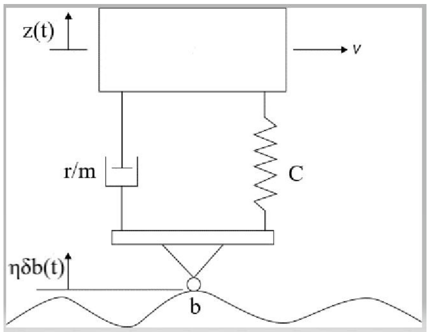

Figure 7: Translated lightly damped harmonic oscillator.

For the evaluation of the impulsive response $z(t)$ of the translated lightly damped harmonic oscillator, the initial boundary value conditions for $\alpha < \omega_o$ acceptable to the corresponding translated forcing function are:

i. When the damped harmonic oscillator is passing through point b time t is measured from this instant and at time $t = b$ the impulsive response becomes zero $z(b) = 0$.

ii. For an instant $t = b^+$, which is the time just after the application of the impulsive force $\eta\delta_b(t)$, the velocity of the damped harmonic oscillator becomes maximum and assumed to be $z'(b^+) = v_o$.

The equation for the displacement of the translated lightly damped harmonic oscillator is same that is for the response of the translated lightly damped harmonic oscillator therefore,

$$d_t(t) = \frac{v_o e^{-\alpha(t-b)}}{\sqrt{\omega_o^2 - \alpha^2}} \sin\sqrt{\omega_o^2 - \alpha^2}(t-b) \quad (36)$$

Where $d_t(t)$ represents displacement for the translated lightly damped harmonic oscillator that is lightly damped because $\alpha < \omega_o$ that refers the impulse damping forcing function z_b is too small as compare to natural frequency ω_o. The displacement is oscillatory function that involves sinusoidal function in its expression and the term $\frac{v_o e^{-\alpha(t-b)}}{\sqrt{\omega_o^2 - \alpha^2}}$ is the amplitude A_t of the translated lightly damped harmonic oscillator that is reducing exponentially with the passage of time varying function, so another form of above expression of displacement is,

$$\vec{d}_t(t) = A_t \sin\sqrt{\omega_o^2 - \alpha^2}(t-b).$$

$$\vec{v}_t'(t) = -\alpha\vec{d}_t(t) + v_o e^{-\alpha(t-b)} \cos\sqrt{\omega_o^2 - \alpha^2}\, t \quad (37)$$

The above equation (37) shows the velocity for the translated lightly damped harmonic oscillator. Moreover, it reveals that the velocity the translated lightly damped harmonic oscillator is the combination of two different sinusoidal functions those are the cosine and the same sine function that exists in displacement of the translated lightly damped harmonic oscillator.

$$\vec{a}_t(t) = (2\alpha^2 - \omega_o^2)\vec{d}_t(t) - 2\alpha v_o e^{-\alpha(t-b)} \cos\sqrt{\alpha^2 - \omega_o^2}(t-b) \quad (38)$$

The above equation (38) express the acceleration for the translated lightly damped harmonic oscillator and shows that it is sinusoidal function of the combination of cosine and the same sine function that exists in displacement of the translated lightly damped harmonic oscillator. The acceleration of the translated lightly damped harmonic oscillator is oscillatory function.

$$T.E \approx \frac{1}{2}C\left(\frac{v_o e^{-\alpha(t-b)}}{\sqrt{\omega_o^2 - \alpha^2}}\right)^2 \approx \frac{1}{2}CA_t^2$$

Where $A_t = \frac{v_o e^{-\alpha(t-b)}}{\sqrt{\omega_o^2 - \alpha^2}}$ is the amplitude of the translated lightly damped harmonic oscillator for response $\vec{d}_t(t)$. The above equation describes that total mechanical energy does not conserve for the translated lightly damped harmonic oscillator, rather it reduces with the increment in time variable and it depends upon amplitude also similar to amplitude it has scalar nature. The energy reduction phenomenon happens due to the external damping agency, in this case of oscillatory system that is F_b. This loss of total energy minimizes the amplitude A_t and finally the oscillations of the translated lightly damped harmonic oscillator become disperse.

Conclusion

The decrement in the oscillations was not significant for the smaller damping force function that referred the lightly damped harmonic oscillator. The Dirac delta function that was basically an impulse function had been introduced as the damping force function for the lightly damped harmonic oscillator. In this work, the impulsive response of the lightly damped harmonic oscillator was obtained by utilizing Fourier transform. The theory of this lightly damped harmonic oscillator was extended on an overdamped harmonic oscillator again by utilizing Fourier transform with the identical damping force function (impulse function). From result it is concluded that Fourier transform can be applied to the lightly damped harmonic oscillators to obtain their impulsive responses because of following advantages.

- Repeated use of the initial boundary value conditions, for ordinary differential equation can be overcome easily.
- Classical pen and paper computational procedure for the determination of solutions of the lightly damped harmonic oscillators can be simplified by this approach to reduce the time consuming factor.
- It can deal with translated damping forcing function associating with the lightly damped harmonic oscillators.
- This approach can be extended to overdamped oscillator by simply reversing the relation of frequency and amplitude.

Although there is plenty of literature available related to damped harmonic oscillators but a little work has been done on the utilization of Fourier transform to solve differential equations of damped harmonic oscillators. This approach will prove a good alternative of Classical methods including Laplace transform for Analyzing impulsive responses of damped harmonic oscillators as well as overdamped harmonic oscillators by analyzing its some basic properties, which is often a major area of concern for academic literature. This technique will prove a good tool for engineering students due to its simplicity in understating the implementation of Fourier transform to differential equations.

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