**On the Duality of Hardy Spaces**

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**Abstract****:** We are concerned with the duality of the Hardy spaces of antianalytic functions on the disk is given which generalizes a result of Bukhvalov. So we prove that under the canonical map when admits analytic projections . If be a complex Banach space and is Lebesgue-Bochner space of -valued integrable functions on the circle and its Hardy type subspace Examples are constructed for bad behavior of the analytic projection and of functions in this dual space if does not belong to the well-known UMD class of Banach spaces.

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1. **Introduction**

If be a complex Banach space with dual space and Let be the Lebesgue–Bochner space of valued *p*th power integrable functions on the unit circle with respect to normalized Lebesgue measure on *.* The space in the title of this paper is the vector–valued Hardy space. where denotes the th Fourier coefficient of . As one might expect, can also be realized, via Poisson integral, as a closed subspace of the Hardy space

( denotes the unit disc). ( These spaces "coincide" iff has the "analytic Radon–Nikodým property" a RNP, see 2.7). The analogous spaces of -valued antianalytic (resp. harmonic) functions are denoted by See Section 2 for details.

It is classical that in the scalar theory the dulality holds; more precisely: the canonical map

is an isomorphism [17,7.2]. The crucial ingredient in the proof is the boundedness of the "analytic (or Riesz) projection" which assigns to the function the function Thus, if is a Banach space such that the analogous map is defined and bounded, the isomorphism holds, and the converse is also true (4.5). This was observed first by Bukhvalov [8] and rediscovered in [33]. Unfortunately, the class of Banach spaces admitting this analytic projection is very restrictive: It coincides with the well-known class *UMD* which is, e.g., smaller than the class of superreflexive Bacach space (see 3.3). The question thus arises how to describe for a general Banach space . In this paper, is represented as a certain space of analytic -valued functions on the disc (4.3, 4.4); this space contains as a weak\* sequentially dense subspace. In general is in neither dense nor closed (4.7): e.g., sufficient for denseness is the Radon-Nikodym property of ; in present of a *RNP* it is also necessary.

The connection of this description of with the analytic projection is still very close: consist exactly of the antianalytic projections of functions in (Corollary 4.5). (Note that for any harmonic function the analytic projection can always be defined, see 3.1). It must be said, however, that the norm of a member depends so explicitly on its action as a functional on that the representation theorem can not be regarded as really satisfactory. My justification is, first that the function is ─which, anyhow, are the functional on ─can enjoy a rather unwieldy boundary behavior. Even if has the *RNP* (in which case is the limit in norm of the, see 4.7). Cf. the discussion in 4.8. This is exemplified by the examples living in , (predual of JT [46]), . These constructions might be of interest for other vector- valued Hardy space or harmonic analysis as well. Second, the result of Bukhavlov mentioned above subordinates itself naturally under the representation given here (4.5) and, anyway, some assertions about the position of in can be made (4.6, 4.7).

I am not treating the case , since this has been done, for several variants of vector-valued spaces, by Blasco [3] and Bourgain [5].

The organization of this paper is as follows. In Section 2, we collect preliminaries on vector-valued Hardy spaces. In section 3, the analytic projection operators are introduced, and first example is given of bad behavior of this operation outside the UMD class (3.5). The representation of described informally above is carried out in Section 4, consider under the canonical map when admits analytic projections . Last but not least, the construction of (counter-) examples fills Section 5.

1. **Preliminaries on vector-valued Hardy spaces**

We begin by recalling some necessary definitions and notation on spaces of integrable functions.

In this paper, is the unit disc in the unit circle, normalized Lebesgue measure on . All spaces of integrable functions will be taken with respect to , which is therefore suppressed innotation. is the indicator function of denote complex Banach spaces, the duall space of (resp. ) the closed (resp. open) unit ball of . The term “isometry” does not include surjectivity, whereas "isomorphism" does. If is an -valued and an -valued function, stands for the scalar function has it usual meaning and is abbreviated as

The basic theory of the Bochner integral and Bochner-Lebesgue spaces is supposed to be known [15]. We are going to explain the less familiar notion of Gel'fand integral and the spaces first. Unless stated otherwise, .

**Definition 2.1.** A function is called scalarly integrable (w.r.t.) if the function is integrable for all in this case, for any Borel set , the Gel'fand integral is well–defined by the formula

[15].

The symbol will often be suppressed.

Now recall that the Banach lattice is order complete [28]; i.e., every order–bounded subset of has a supremum in (supremum in the sense of order in , denoted by -sup). Put

Following Bukhvalo [8], one defines for the function and the semi-norm The null space of the semi-norm on is easily recognized as (Note that in the formulation "” the exceptional null set depends on . Typical example: the th unit vector. We have, , but

Finally, put with the associated norm For and are well-defined, since independent of the choice of representative. Obviously, we have as a closed subspace (i.e., the canonical map is an isometry.

Now let It is not hard to see that is a (well-defined!) member of satisfying ,

so that acts as a bounded linear functional of norm at most , on . The importance of the space lies in the fact that it is the exact dual space of

**Theorem 2.2.** Let be a Banch space, the map

is a (well-defined) isometric isomorphism.

It is in a canonical sense equivalent [21] to the perhaps more widespread representation of using the upper integral [24].

**Lemma 2.3.** For , we denote by  the *n*th Fourier coefficient and by the Poission integral of . (The integrals, of course, Gel’fand integrals.) Here is the Poisson kernel,

An easy computation yields, as in the scalar case,



with absolutely and in locally uniformly convergent series.

Now let be a Banach subspace. The following conditions on are equivalent [21] :

Borel;



 .

The space of those is denoted by Obviously  as closed subspaces.

An important feature of functions in that is, of the strongly measurable members of is exhibited in the following.

**Theorm 2.4.** Let  . Then in

**Proof**. Since translation in the argument of a function is a continuous map  [23] (here the Bochner integrability enters), the scalar proof can be carried over without difficulty [8].

**Corollary 2.5.** Trigonometric polynomials   are dense in and in .

**Proposition 2.6.** For we denote by , 

Cauchy integral of .Here is the Cauchy kernel, is an analytic valued function with Taylor series ; in particular, if , where is a Banach subspace. Comparing the coefficients of  and  yields: is analytic  Any with these properties is called "of analytic type " ("analytic" for short). The (obviously closed) subspace of analytic members of is donted by of course,

As a corollary of Theorem 2.4. "analytic" polynomials  are dense in

**Proposition 2.7.** We define  harmonic  where  and . A word on the notion of a Banach space valued harmonic function seems in order. Exactly as in the better known case of holomorphic functions [23] any two reasonable definitions of harmonicity for a Banach space valued function are equivalent. To be more specific, any of the following conditions on  implies all the others [21] .

1. is strongly harmonic, i.e, and .
2. is weakly harmonic, i.e. is harmonic
3. (If ) is weak\* harmonic, i.e., is harmonic 
4. such that in with absolutey and locally uniformly convergent series.

By the usual sub harmonicity argument it is easily proved that increases with for harmonic, [21]. Also, one has the scale if the inclusions are of norm

The following Poisson integral representation theorem [21Theorem (1.5)] is essentially a concise formulation of results of Grossetete [18, Sect. l] and Bukhvalov [8, Theorem 2.3]. Let as in Lemma 2.2. (e.g.,

.

**Theorem 2.8.** The Poisson integral defines

an isometry 

an isometric isomorphism if 

For the sake of clarity, I remark that the isometry in  is never subjective (except  *).* The full representation space for would be , the space of valued vector measures with bounded variation on the Borel sets of The space appearing above corresponds via the identification exactly to the subspace consisting of 𝜆-absolutely continuous members of , this is essentially the "generalized theorem of Lebesgue Nikodym"[16].

**Definition 2.9.** Thus is a closed subspace and isometrically; for with norm .

Function in these spaces behave well as regards boundary values:

**Proposition 2.10.** If with then a.e. Conversely, if and 

exists a.e., then  and  Summing up,

**Proposition 2.11.** The analytic vector–valued Hardy spaces are defined in the range as

,

Thus, of course, is analytic}, We also define the vector–valued Nevanlinna class



Again, the suprema are increasing limits as and we have the scale .

if

the first inclusion is because for analytic (use Jensen's inequality), the other inclusions are clearly of norm . We will make use of the following result due to Danilevich [14] in a more general Frechet space setting. For a simpler proof in the Banach space context [21].

**Proposition 2.12.** Let be a separable Banach space and  Then exists a.e in 

Returning to the range the following Poisson integral representation theorem is, at least if ,a trial consequence of the preceding one (2.7), by the remarks made in 2.6. It is due to Ryan [34]. Let again .

**Theorem 2.13.** For the Poisson (or Cauchy) integral defines an isometric isomorphism .

In view of Theorem 2.8 and the remarks following it, the theorem for is tantamount to the knowledge that every “analytic” member of is already in , i.e., the vector-valued and . Riesz theorem [18, 2. Corollary; 21, Theorem (2.3); 25, p. 316; 35, Theorem l] which in turn is a trivial consequence of the scalar-valued one.

**Definition 2.14.** 

Thus is closed subspace and for

As one might expect, the assertions of Preposition 2.7 hold for in the full range that is, [11]. In most of what follows, we will identify the spaces and , more precisel y with and with its boundary value .

**Proposition 2.15.** Bukhualov and Danilevich were the first to recognize the close connection between the Radon-Nikodym property [15] and the theory of valued spaces. Their result may be summarized as follows: has iff (for one (all) various extensions of this theorem, as regards the extreme values of, have been given independently by Blasco [2] and the author [21]. I state here only what is needed later.

**Theorem 2.16.**  has the *RNP* iff (that is, by Proposition 2.7, iff every bounded harmonic function has radial limits ).

**Theorem 2.17.** A Banach space has the analytic Radom-Nikodym property if the following equivalent properties are satisfied:

(that is by Proposition 2.11., every bounded analytic function has radial limits a.e.).

For all that is by Proposition 2.11, every has radial limits a.e.).

Every has radial limits a.e.

**Proof.** It obviousty suffices to show . But this follows trivially from the vector–valued . and . If then with without zeroes.

For example, does not have a : Consider  . It is also clear from the above that *RNP* implies a *RNP*. The converse is not true; an example is provided by the space which has a *RNP*, as does every Banach lattice not containing . This major result is again due to Bukhvalov and Danilevich [11], for a simplified proof using semi-embedding).

1. **Analytic projection**

As in the scalar-valued case, the analytic (or Riesz) projection is intimately connected with the description of duals of Hardy spaces. Let 

**Definition 3.1.** For a harmonic function with series let be the analytic function is called the analytic projection of .

Thus, for for simplicity, this will often be abbreviated to there may or may not be a with (equivalently, with formal Fourier series if there is, this (necessarily unique) is also dented by and called the analytic projection of for example, the analytic projection of a trigonometric polynomial is



For technical reasons, the antianalytic projection, denoted by  will also be used: For as above, 

, etc. (Here denotes convolution with the complex conjugate of the Cauchy kernel.) It is the "adjoint" of the analytic projection in the sense that, e.g., for trigonometric polynomials 



**Definition 3.2.** For we say " admits analytic" projection if  is a bounded operator . Equivalent conditions are is a bounded operator (or, by denseness, only on the trigonometric polynomials); alternatively: is a bounded operator . One can also show[22] that it is the same to demand that is a bounded operator , or that is complemented in . By duality (see Definition 3.1), admits analytic projection iff admits [anti-] analytic projection [8].

**Lemma 3.3.** That boundedness of the analytic projection is equivalent to boundedness of the Hilbert transform where a.e. .

Superreflexivity of is derived already from -boundedness of the -valued Hilbert transform on the circle (=conjugate function operator, which is trivially equivalent to the analytic projection, too). In a similar vein, we have.

**Proposition 3.4.** Suppose for all . Then a *RNP* implies *RNP* for .

**Proof.** To derive *RNP* for , one has to show that every has radial limits a.e. (Proposition 2.15.) Putting , so that as well, one easily obtains

.

By assumption, , and if has a it follows that have radial limits a.e. (2.17), whence the same holds for .

**Example 3.5.** The proposition says in other words that if then analytic projection cannot map into . Moreover, the proof tells one how to produce examples: Take any without a.e. existing boundary values, then necessarily .

As a concrete example, consider and is harmonic, e.g., by condition (ii) of 2.7, and  for all , thus .

Since the series expansion of is , where , we have 

so that



does not depend on  and as



so that indeed .

Keeping  fixed, we will show now that analytic projection is not a bounded operator in the sense that 

Take as above and, for  put then  with for all . On the other hand, as is easy to see, Thus, as computed above,



hence

This means as asserted.

As a corollary, analytic projection is not a bounded operator (and thus; by the closed graph theorem, not an operator at all) for any , or, what amounts to the same, it is not -bounded on the -valued trigonometric polynomials. Note that even for this does not follow directly from the result about superreflexivity quoted in Lemma 3.3, since the first part of its proof, proceeding along the lines of [31, 23.] works with step functions and thus outside For further examples of bad behaviour of the analytic projection see Examples 5.2,5.3&5.4.

1. **The Dual Space of**

Let be a complex Banach space and Recall the identifications

We Define

(Obviously, on the disc we have via Poisson integral

The spaces are defined analogously, namely as (resp. .

**Remark 4.1.** By general Banach space theory, Where is the annihilator of In In 2.1, was identified as , and is easily recognized as , since analytic polynomials are dense in . We arrive at the description

(canonically isometrically isomorphic), but of course one aims at a description of as a space of functions, not equivalence classes.

Consider the canonical injective operators .

which is the composition .

If is a *UMD* space, then is an isomorphism, since is then given by the antianalytic projection modulo its kernel Vice versa, if is an isomorphism , it is immediate to verify that

is the antianalytic projection. We arrive at a theorem of Bukhvalov [8]: canonically admits analytic projection admits [anti-] analytic projection i.e., (see 3.2, 3.3).

The scalar multiplication in is to be understood as [37]. This makes the dual pairing sesquilinear and allows one to replace by in all of these consideration [8]. Alternatively, the latter effect could also be achieved by giving the dual pairing , defined as here and in [8] the new meaning as in [10], similar to the case of Bergman spaces in [9].)

The problem arises to describe for a general Banach space as a space of functions–the more, since the ***UMD*** condition on is extremely restrictive. The description (4.6) of as , a space of antianalytic - valued functions on the disc, is an attempt in this direction. Since the norm of depends rather explicitly on action as a functional or , this answer is not really satisfactory. For instance, in the concrete case it does not yield an illuminating description of but this might well be in the nature of things because of the bad behavior of - valued analytic projection exhibited in Example 5.4. On the other hand, Bukhvalov's theorem mentioned above subordinates itself in a natural way as a special case (4.5), and some assertions about the position of in can be made (4.10, 4.12).

In what follows, for a function on and denotes function on and/or on . If is defined on (and makes sense ), means .

**Lemma 4.2.** Let harmonic with corresponding series expansions

In particular, if and then .

**Proof.**  with uniformly convergent series on ( is fixed). Thus

The other equality is proved in the same way.

**Corollary 4.3.** Let harmonic,

**Proof.** apply the lemma to

Follows from the lemma and ; the latter because in (2.3).

Part 2 of this corollary says, in other words, that as weak\* in a fact which also follows directly from the general theory of Poisson integral representation [8].

**Definition 4.4.** Let

(Note that is in thus in after the discussion in 4.1)

**Remarks 4.5.** Let be antianalytic then:

with is a normed space (completeness will follow later).

increases to as .

Where is a constant independent of (and ), and thus

the first inclusion being continuous.

in particular:

increases to as in particular, for

**Proof.**  If at all only requires proof. means in hence , in (all ), the canonical map being injective. Hence .

Take arbitrary , By Lemma 4.2,





First inequality: fix of course, by scalar theory (or the discussion in 4.1). Hence ( cf. [17, p. 113] )







Now let

Second inequality:

By Lemma 4.2, if

On the other hand, by Lemma 4.2,

Apply to ; then .

**Corollary 4.6.** Let , be a sequence where be antianalytic then,

1. with .
2. increases to

where .

such that is independent of .

Hence an antianalytic and

1. implies that



For

implies that 

increases to as

For





**Proof.** i. If  then If it means that in hence in and the map is injective.

That is  for every and i.e. .

ii. For Corollary 4.3, shows that

**Theorem 4.7.** The map where is a (well-defined) isometric isomorphism.

**Proof.** First of all, for , by Remark 4.5,



sincein as noted earlier. Thus we can dispose of the version.

Now fix . For distinction, the functional on (earlier identified with ) will be denoted by We have

If *f* is an analytic monomial  then exists:

After Remark 4.5, Whence

 as 

Since analytic monomials form a total subset of (2.6), exists for all and

If the calculation above yields hence in . This proves injectivity.

Surjectivity and other estimates: Let be given. Choose a Hahn-Banach extension ; by 2.1, is given by:put the antianalytic projection of . For , by 4.2 and 4.3,

Thus represents and

which completes the proof.

In particular, is a Banach space. In terms of the canonical isometric isomorphism (4.1), the proof yields.

**Note** 4.8. If , then defines the functional on . On the other hand, by Remark 4.4, , as well and thus defines, after the theorem, the functional . Fortunately, these two coincide, by Corollary 4.3, .

**Corollary 4.9.** is an isometric isomorphism. In particular, consists exactly of the antianalytic projections of functions in (Here denotes equivalence class mod

I want to show now that Bukhvalov’s theorem already derived in Section 4.1 is contained in Theorem 4.7:

**Corollary 4.10.** (Bukhvalov) under the canonical map (see 4.1) iff admits analytic projection .

**Proof.** In view of Theorem 4.7 (and note), (canonically) iff as spaces of functions on the disc, with (then automatically (4.3, ) equivalent norms Suppose this holds and let . For any trigonometric polynomial , whence





(last equality because is of antianalytic type), so that admits analytic projection . Conversely, if this latter condition is fulfilled with norm , say, then for any antianalytic,

 



so that (with equivalent norms).

**Corollary 4.11.**  under the canonical map when admits analytic projections .

**Proof.** Theorem 4.7 can show that if and only if as spaces of functions on the disc with equivalent norm Now let for any trigonometric polynomial sequence where







Then admits analytic projections .

Hence for any  antianalytics, then











where is a norm, so that

I continue with some assertions about the position of in As regards the weak\* topology,

**Proposition 4.12.**  If then in the weak\* topology and as

Antianalytic polynomials are weak\* sequentially dense in What is more, sequentially dense in .

**Proof.**  Clear by Note 4.8 and Remark 4.5, .

Note that weak\* denseness alone of antianalytic polynomials in would follow already from the "abstract" criterion: a Banach (or locally convex) space, a vector subspace, then is weak\* dense in iff

Put here {antianalytic polynomials}.

To prove (the second assertion of) take and choose a sequence then weak\* and Put then also weak\* and i.e., This is a open set in , because the inclusions are continuous by 4.5, . Since antianaltyic polynomials are dense in , we can choose one, say in with For we have



so that weak\* in as well.

**Corollary 4.13.** -valued antianalytic polynomials, equipped with , norm that is ,

, .

As regards the norm topology, we have

**Theorem 4.14.** If has RNP, then for all .

The following are equivalent:

(a) has *RNP*

(b) has a *RNP* and is dense in

The following are equivalent:

(a)

(b) is closed in

(c) is closed in

**Proof.** By Corollary 4.7, the antianalytic projection is a bounded surjective operator [36] but if has RNP, then.

Now fix take any with , and use that in . (2.3). It follows that 

(a)(b) Follows from (b)(a) by Corollary 4.7, we can identify with The density assumption then says that the canonical map has dense image. Since has a *RNP*, we have (2.17) and it is clear that the map is an isometry.

It follows that has dense image as well and is thus surjectve . This means

so that has *RNP* [36].

are trivial. let be a constant such that over . For a trigonometric polynomial  whence by Corollary 4.12,



exactly as in the proof of Corollary 4.10, which also proves now (a)

**Remarks 4.15.** (a) Part shows that is in general not dense in , e.g., certainly not if if 

(b) In other words, the canonical map in general does not have dense image. In contrast to this, the analogous map always has dense image--it contains all equivalence classes (mod of --valued trigonometric polynomials.

(c) In the proof of , we have had and it was therefore trivial that is an isometry. I claim that this is always true, i.e., without the a assumption on :

Take . since in after 2.3, one can write

The reverse inequality being trivial, the claim is proved.

(d) Combining (b) and (c) yields: is dense in iff is an isometric isomorphism.

Since in general, and even functions on the disc posses "boundary values" on the circle only in a very weak sense, not much can be expected about boundary values of functions. Anyway, if then with radial limit function for all I will pursue the question if this collection of functions give rise to single function with the property that for all a.e. (the exceptional set allowed to vary with ). (Of course, if then its "boundary value") the unique with does this job. But for a general such a automatically scalarly measurable w.r.t. might exist without being in . The remote aim of this attempt would be, of course, to replace the action of the functional as by a single integral .

After Corollary 4.8, Fix Then, for any function the condition a.e. is equivalent to saying a.e., where the last equality sign identifies the scalar function with its boundary value.

In the following examples , it will be shown that, even for such a function need not exist. In these examples, In the first one, is even strongly measurable, that is, Since has **RNP**, thus this is naturally also the case in the last example. What makes this one more interesting is the fact that, due to the **RNP** of and Theorem 4.14, strongly in for all and the boundary behavior of can still be as bad as it can be.

It is of course equivalent to construct these examples with the analytic instead of the antiana1lytic projection.

1. **Examples**

**Lemma 5.1.** For and :



**Proof.** This is an elementary calculation and, of course, well known.

I need some notation. The infinite dyadic tree is denoted by [27].

For put so that is theth dyadic interval of the th generation. A number is called dyadic if it is of the from for some . For non–dyadic let be the "branch" of the tree associated with Obviously, For put

**Example 5.2.** There is such that.

1. exists for in . In particular, because of Proposition 2.12;

(b) there exists no function with the property a.e.

**Proof .** I realize as and denote it again by Let be a positive null sequence, which will be specified later.

Put By the Pettis measurability theorem [15], By Lemma 5.1,

so that for non–dyadic,

the limit taken coordinate-wise. (if is dyadic, the coordinate-wise radial limit does not exist). To prove (a) and (b), it suffices to choose in such a way that this last tuple does not blong to for all (non-dyadic) has to be independent of , of course.) Now fix non-dyadic. Since always one only has to estimate , or the same expression only along . but for

e.g., for

The next example lives in the canonical predual of the James tree space [27]. Since there is no lack of examples in more elementary Banach spaces.

**Example 5.3.** (B). There is such that

(a) exists for no in in particular, because of Proposition 2.12 (note that and that is separable [27] );

(b) there exists no function with the property a.e.

**Example 5.4. .** There is such that

(a) exists for in . In particular, because of Proposition 2.12;

(b) there exists function with the property a.e.

**Proof .** I realize as and denote it again by . Let be a positive summable sequence, which will be specified later.

Put It is clear that is really in for all and that Moreover, is strongly measurable by Pettis theorem [15]; that is

By Lemma 5.1,

So that for non–dyadic ,

(If is dyadic, the coordinate-wise radial limit does not exist.) To prove (a) and (b), it suffices to choose in such a way that this last tuple does not belong to for all (non-dyadic) . This will be achieved through the following.

**Lemma 5.5.** For non-dyadic,

.

Accepting the lemma for a moment, we conclude as follows: fix non-dyadic,







e.g, for

**Proof.** W.l.o.g., then putting gives . Now









since .

Note. Would we not have given away half of the terms in the first estimate, we could achieve the (irrelevant) improvement .

**Remark 5.6.** Let  be the function just constructed. Since and is coordinate- wise real, we have ,

and is the function just computed. Bearing in mind that, by Corollary 4.9, there seems to be little hope for a simple description of

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