# Indirect Boundary Element Technique For The Flow Past A Fixed Sphere

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**Abstract:** In this paper, an indirect boundary element technique is applied for calculating the flow past (i.e. velocity distribution) a fixed sphere. Such technique requires the concept of sources or doublets, which is simple and flexible. A comparison study between computed and analytical results is also made and it can be seen from tables and figures that the computed results are good in agreement with analytical results for a fixed sphere. [ Journal of American Science 2009, 5(8):68-73].(ISSN:1545-1003).

**Keywords:** Indirect boundary element technique, Flow past, Velocity distribution, Fixed sphere.

## 1. Introduction

From the time of fluid flow modeling, it had been struggled to find the solution of a complicated system of partial differential equations (PDE) for the fluid flows which needed more efficient numerical methods. With the passage of time, many numerical techniques such as finite difference technique, finite element technique ( Hirt et al,1978, Markatos,1983, Demuran et al,1982, Ecer,1982) and boundary element technique etc. came into beings which made possible the calculation of practical flows. Due to discovery of new algorithms and faster computers, these techniques were evolved in all areas in the past. Such techniques are CPU time and storage hungry. The term 'boundary elements' originated in the department of civil engineering, Southampton University, United Kingdom (Brebbia,1978). One of the advantages is that with boundary elements one has to discretize the entire surface of the body, whereas with domain techniques it is essential to discretize the entire region of the flow field. The most important characteristics of boundary element technique are the much smaller system of equations and considerable reduction in data, which is prerequisite to run a computer program efficiently. Furthermore, this technique is well–suited to problems with an infinite domain. From above discussion, it is concluded that boundary element technique is a time saving, accurate and efficient numerical technique as compared to other numerical techniques which can be classified as direct and indirect boundary element techniques. The indirect technique utilizes a distribution of singularities over the boundary of the body and computes this distribution as the solution of integral equation. The equation of direct boundary element technique can be formulated using either as an approach based on Green's theorem (Lamb,1932, Milne-Thomson,1968, Kellogg,1929) or a particular case of the weighted residual formulation (Brebbia and Walker,1980). The flow field caculations around three-dimensional bodies were calculated (Hess and Smith,1962;1967). Direct boundary element technique was applied for potential flow problems ( Morino et al,1975). Boundary element technique is now widely applied to several fields such as potential theory, elasticity, elastostatics, elastodynamics, etc ( Brebbia and Walker, 1980) and biomedical problems (Muhammad et al,2009). Thus boundary element technique s are more powerful than other numerical techniques because of their boundary modeling approach.

**Flow Past a Fixed Sphere:**

Let a fixed sphere be of radius 'a' with center at the origin and let the onset flow be the uniform stream of velocity U in the positive direction of the y–axis, as shown in figure (1).

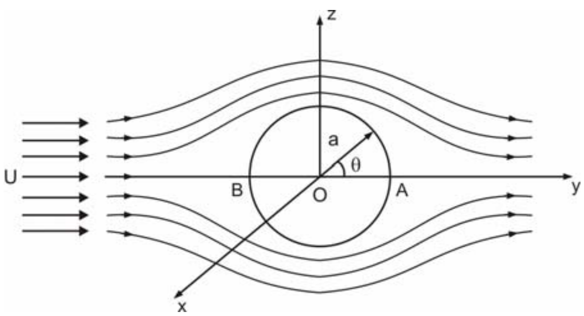


Figure 1

The velocity potential and stream function for a sphere of radius $a$ moving of velocity $U$ in the negative direction of y – axis in an infinite mass of fluid at rest at infinity are given by (Shah, 2008)

$$\phi = -\frac{1}{2} U \frac{a^3}{r^2} \cos\theta , \quad \psi = \frac{1}{2} U \frac{a^3}{r} \sin^2\theta \quad (1)$$

If a uniform velocity field 'U' is impressed upon the sphere and the fluid in the positive direction of y – axis , the sphere will come to rest and the uniform stream which was at rest at infinity will start moving with velocity $U$ in the positive direction of y – axis . Thus the streaming motion past a fixed sphere will be obtained.

The superposition of the velocity field $U$ in the positive direction of y – axis amounts to the addition of the term $-U r \cos\theta$ to the velocity potential and the term $-\frac{1}{2} U r^2 \sin^2\theta$ to the stream function in equations (1) . Thus the velocity potential and stream function for the streaming motion past a fixed sphere of radius $a$ in the positive direction of y – axis take the form

$$\begin{aligned} \phi &= -U r \cos\theta - \frac{1}{2} U \frac{a^3}{r^2} \cos\theta \\ &= -U\left(r + \frac{a^3}{2r^2}\right)\cos\theta \qquad (2) \\ \psi &= -\frac{1}{2} U r^2 \sin^2\theta + \frac{1}{2} U \frac{a^3}{r} \sin^2\theta \\ &= -\frac{1}{2} U \left(r^2 - \frac{a^3}{r}\right)\sin^2\theta \qquad (3) \end{aligned}$$

The velocity components at any point $(r, \theta)$ are given by

$$v_r = -\frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta$$

$$v_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = -U\left(1 + \frac{a^3}{2r^3}\right)\sin\theta$$

The speed at any point in the flow field is given by

$$\begin{aligned} V &= \sqrt{v_r^2 + v_\theta^2} \\ &= U \sqrt{\left(1 - \frac{a^3}{r^3}\right)^2 \cos^2\theta + \left(1 + \frac{a^3}{2r^3}\right)^2 \sin^2\theta} \qquad (4) \end{aligned}$$

The velocity components at any point $(a, \theta)$ on the boundary of a sphere $r = a$ become

$$v_r = 0, \quad v_\theta = -\frac{3}{2} U \sin\theta \qquad (5)$$

Equation (5) shows that the velocity on the boundary of the sphere is purely tangential .

Therefore the speed at any point on the sphere itself is given by

$$\begin{aligned} V &= \sqrt{v_r^2 + v_\theta^2} \\ &= \sqrt{0 + \left(-\frac{3}{2} U \sin\theta\right)^2} \\ &= \frac{3}{2} U \sin\theta \qquad (6) \end{aligned}$$

For exterior flow for three-dimensional problems, the mathematical formulation for indirect boundary element method in terms of doublets distribution over the boundary S of the body is given by

$$-\frac{1}{2}\Phi_i + \phi_\infty + \iint_{S-i} \Phi \frac{\partial}{\partial n}\left(\frac{1}{4\pi r}\right) dS = y_i \qquad (7)$$

Which is discretized by dividing the boundary of the body under consideration into 'm' elements and finally, it is written in matrix form as

$$[H]\{\underline{U}\} = \{\underline{R}\} \qquad (8)$$

Whereas usual [H] is a matrix of influence coefficients, $\{\underline{U}\}$ is a vector of unknown total potentials $\Phi_p$ and $\{\underline{R}\}$ on the R.H.S. is a known vector whose elements are the negative of the values of the velocity potential of the uniform stream at the nodes on the boundary of the body.

## 3. Boundary Conditions

The boundary condition to be satisfied over the surface of a sphere is

$$\frac{\partial \phi_{sphere}}{\partial n} = U(\hat{n} \cdot \hat{j}) \qquad (9)$$

where $\phi_{sphere}$ is the perturbation velocity potential of a sphere and $\hat{n}$ is the outward drawn unit normal to the surface of a sphere

Let $f(x, y, z) = x^2 + y^2 + z^2 - a^2$

Then $\nabla f = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

Therefore $\hat{n} = \dfrac{\nabla f}{|\nabla f|} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{(2x)^2 + (2y)^2 + (2x)^2}}$

Thus $$\hat{n} \cdot \hat{j} = \frac{2y}{\sqrt{(2x)^2 + (2y)^2 + (2x)^2}}$$

$$= \frac{y}{\sqrt{x^2 + y^2 + z^2}}$$

The equation of the surface of the sphere is

$x^2 + y^2 + z^2 = 1$, where the radius $a$ is taken as 1.

Therefore, the boundary condition in (9) takes the form

$$\frac{\partial \phi_{sphere}}{\partial n} = U y$$

$$= y, \quad (\text{Taking } U = 1) \qquad (10)$$

Equation (8) is the boundary condition which must be satisfied over the boundary of a sphere.

## 4. Discretization of Elements

Consider the boundary of the sphere in one octant to be divided into three quadrilateral elements by joining the centroid of the boundary with the mid points of the curves in the coordinate planes as shown in figure (2) (Mushtaq et al,2008).

Then each element is divided further into four elements by joining the centroid of that element with the mid–point of each side of the element. Thus one octant of the boundary of the sphere is divided into 12 elements and the whole bounday of the body is divided into 96 boundary elements. The above mentioned method is adopted in order to produce a uniform distribution of element over the boundary of the body.

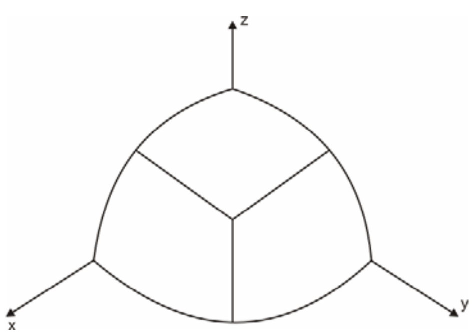


Figure 2

Figure (3) shows the method for finding the coordinate ($x_p$, $y_p$, $z_p$) of any point P on the surface of the sphere.

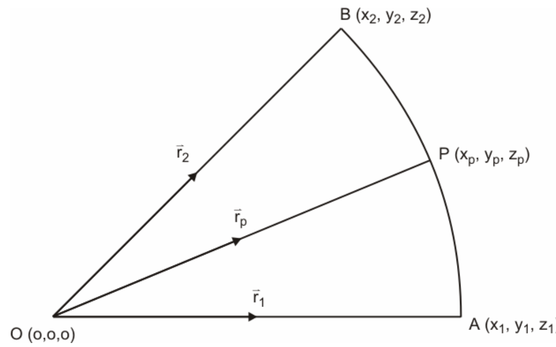


Figure 3

From figure (3), we have the following equation

$$|\vec{r}_p| = 1$$

$$\vec{r}_p \cdot \vec{r}_1 = \vec{r}_p \cdot \vec{r}_2$$

$$(\vec{r}_1 \times \vec{r}_2) \cdot \vec{r}_p = 0$$

or in cartesian form

$$x_p^2 + y_p^2 + z_p^2 = 1$$

$$x_p (x_1 - x_2) + y_p (y_1 - y_2) + z_p (z_1 - z_2) = 0$$

$$x_p (y_1 z_2 - z_1 y_2) + y_p (x_2 z_1 - x_1 z_2) + z_p (x_1 y_2 - x_2 y_1) = 0$$

As the body possesses planes of symmetry, this fact may be used in the input to the program and only the non–redundant portion need be specified by input points. The other portions are automatically taken into account. The planes of symmetry are taken to be the coordinate planes of the reference coordinate system. The advantage of the use of symmetry is that it reduces the order of the resulting system of equations and consequently reduces the computing time in running a program. As a sphere is symmetric with respect to all three coordinate planes of the reference coordinate system, only one eighth of the body surface need be specified by the input points, while the other seven–eighth can be accounted for by symmetry.

The calculated velocity distributions are compared with analytical solutions for the sphere using Fortran programming.

**Table 1: Comparison of the computed velocities with exact velocity over the surface of a sphere using 24 boundary elements.**

| ELEMENT | XM | YM | ZM | COMPUTED VELOCITY | EXACT VELOCITY |
|---|---|---|---|---|---|
| 1 | -.321E+00 | -.748E+00 | .321E+00 | .13129E+01 | .13953E+01 |
| 2 | -.748E+00 | -.321E+00 | .321E+00 | .63485E+00 | .77853E+00 |
| 3 | -.748E+00 | .321E+00 | .321E+00 | .63485E+00 | .77853E+00 |
| 4 | -.321E+00 | .748E+00 | .321E+00 | .13129E+01 | .13953E+01 |
| 5 | .321E+00 | .748E+00 | .321E+00 | .13129E+01 | .13953E+01 |
| 6 | .748E+00 | .321E+00 | .321E+00 | .63485E+00 | .77853E+00 |
| 7 | .748E+00 | -.321E+00 | .321E+00 | .63485E+00 | .77853E+00 |
| 8 | .321E+00 | -.748E+00 | .321E+00 | .13129E+01 | .13953E+01 |
| 9 | -.321E+00 | -.321E+00 | .748E+00 | .13129E+01 | .13953E+01 |
| 10 | -.321E+00 | .321E+00 | .748E+00 | .13129E+01 | .13953E+01 |
| 11 | .321E+00 | .321E+00 | .748E+00 | .13129E+01 | .13953E+01 |
| 12 | .321E+00 | -.321E+00 | .748E+00 | .13129E+01 | .13953E+01 |

**Table 2: Comparison of the computed velocities with exact velocity over the surface of a sphere using 96 elements.**

| ELEMENT | XM | YM | ZM | COMPUTED VELOCITY | EXACT VELOCITY |
|---|---|---|---|---|---|
| 1 | -.177E+00 | -.934E+00 | .177E+00 | .14526E+01 | .14747E+01 |
| 2 | -.522E+00 | -.798E+00 | .157E+00 | .12418E+01 | .12623E+01 |
| 3 | -.798E+00 | -.522E+00 | .157E+00 | .82398E+00 | .84609E+00 |
| 4 | -.934E+00 | -.177E+00 | .177E+00 | .34768E+00 | .38819E+00 |
| 5 | -.934E+00 | .177E+00 | .177E+00 | .34768E+00 | .38819E+00 |
| 6 | -.798E+00 | .522E+00 | .157E+00 | .82398E+00 | .84609E+00 |
| 7 | -.522E+00 | .798E+00 | .157E+00 | .12418E+01 | .12623E+01 |
| 8 | -.177E+00 | .934E+00 | .177E+00 | .14526E+01 | .14747E+01 |
| 9 | .177E+00 | .934E+00 | .177E+00 | .14526E+01 | .14747E+01 |
| 10 | .522E+00 | .798E+00 | .157E+00 | .12418E+01 | .12623E+01 |
| 11 | .798E+00 | .522E+00 | .157E+00 | .82398E+00 | .84609E+00 |
| 12 | .934E+00 | .177E+00 | .177E+00 | .34768E+00 | .38819E+00 |
| 13 | .934E+00 | -.177E+00 | .177E+00 | .34768E+00 | .38819E+00 |
| 14 | .798E+00 | -.522E+00 | .157E+00 | .82398E+00 | .84609E+00 |
| 15 | .522E+00 | -.798E+00 | .157E+00 | .12418E+01 | .12623E+01 |
| 16 | .177E+00 | -.934E+00 | .177E+00 | .14526E+01 | .14747E+01 |
| 17 | -.157E+00 | -.798E+00 | .522E+00 | .14495E+01 | .14801E+01 |
| 18 | -.470E+00 | -.703E+00 | .470E+00 | .13038E+01 | .13113E+01 |
| 19 | -.703E+00 | -.470E+00 | .470E+00 | .96588E+00 | .10301E+01 |
| 20 | -.798E+00 | -.157E+00 | .522E+00 | .82398E+00 | .84609E+00 |
| 21 | -.798E+00 | .157E+00 | .522E+00 | .82398E+00 | .84609E+00 |
| 22 | -.703E+00 | .470E+00 | .470E+00 | .96588E+00 | .10301E+01 |
| 23 | -.470E+00 | .703E+00 | .470E+00 | .13038E+01 | .13113E+01 |
| 24 | -.157E+00 | .798E+00 | .522E+00 | .14495E+01 | .14801E+01 |
| 25 | .157E+00 | .798E+00 | .522E+00 | .14495E+01 | .14801E+01 |
| 26 | .470E+00 | .703E+00 | .470E+00 | .13038E+01 | .13113E+01 |
| 27 | .703E+00 | .470E+00 | .470E+00 | .96588E+00 | .10301E+01 |
| 28 | .798E+00 | .157E+00 | .522E+00 | .82398E+00 | .84609E+00 |
| 29 | .798E+00 | -.157E+00 | .522E+00 | .82398E+00 | .84609E+00 |
| 30 | .703E+00 | -.470E+00 | .470E+00 | .96588E+00 | .10301E+01 |
| 31 | .470E+00 | -.703E+00 | .470E+00 | .13038E+01 | .13113E+01 |
| 32 | .157E+00 | -.798E+00 | .522E+00 | .14495E+01 | .14801E+01 |

| | | | | | |
|---|---|---|---|---|---|
| 33 | -.157E+00 | -.522E+00 | .798E+00 | .14495E+01 | .14801E+01 |
| 34 | -.470E+00 | -.470E+00 | .703E+00 | .13038E+01 | .13113E+01 |
| 35 | -.522E+00 | -.157E+00 | .798E+00 | .12418E+01 | .12623E+01 |
| 36 | -.522E+00 | .157E+00 | .798E+00 | .12418E+01 | .12623E+01 |
| 37 | -.470E+00 | .470E+00 | .703E+00 | .13038E+01 | .13113E+01 |
| 38 | -.157E+00 | .522E+00 | .798E+00 | .14495E+01 | .14801E+01 |
| 39 | .157E+00 | .522E+00 | .798E+00 | .14495E+01 | .14801E+01 |
| 40 | .470E+00 | .470E+00 | .703E+00 | .13038E+01 | .13113E+01 |
| 41 | .522E+00 | .157E+00 | .798E+00 | .12418E+01 | .12623E+01 |
| 42 | .522E+00 | -.157E+00 | .798E+00 | .12418E+01 | .12623E+01 |
| 43 | .470E+00 | -.470E+00 | .703E+00 | .13038E+01 | .13113E+01 |
| 44 | .157E+00 | -.522E+00 | .798E+00 | .14495E+01 | .14801E+01 |
| 45 | -.177E+00 | -.177E+00 | .934E+00 | .14526E+01 | .14747E+01 |
| 46 | -.177E+00 | .177E+00 | .934E+00 | .14526E+01 | .14747E+01 |
| 47 | .177E+00 | .177E+00 | .934E+00 | .14526E+01 | .14747E+01 |
| 48 | .177E+00 | -.177E+00 | .934E+00 | .14526E+01 | .14747E+01 |

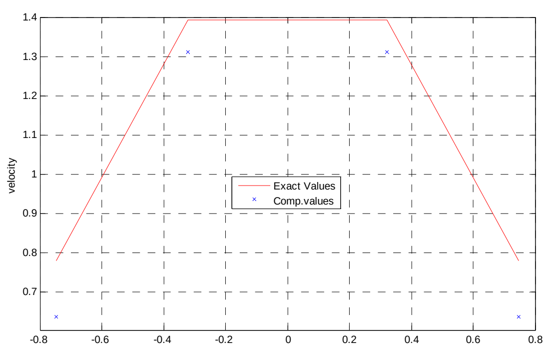


Figure 3: Comparison of computed and analytical velocity distributions over the surface of a sphere using 24 boundary elements

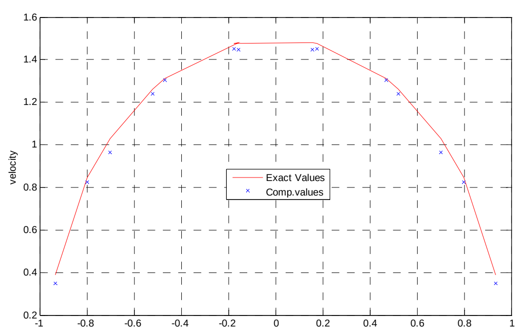


Figure 4: Comparison of computed and analytical velocity distributions over the surface of a sphere using 96 boundary elements

## 5. Conclusion

An indirect boundary element technique has been applied for the calculation of flow past a fixed sphere. The calculated flow velocities obtained using this technique are compared with the analytical solutions for flow over the boundary of a sphere. It is found that the computed results obtained by such technique are good in agreement with the analytical ones for the body under consideration and the accuracy of the results increases with the increase of number of boundary elements.

## Acknowledgements

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