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Studies On Σ-Statistical Convergence And Lacunary Σ-Statistical Convergence

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Abstract: The mappings σ are one-to-one and such that $\sigma^m(k) \neq k$ for all positive integers k and m, where $\sigma^m(k)$ denotes the m^{th} iterate of the mapping σ at k. Thus Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x) = \lim \xi_k$ for all $x \in c$. In case σ is the translation mapping k→k+1, an invariant mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

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1.1 Introduction

Convergence of random variables (sometimes called stochastic convergence) is where a set of numbers settle on a particular number. It works the same way as convergence anywhere else; For example, cars on a 5-line highway might converge to one specific lane if there's an accident closing down four of the other lanes. In the same way, a sequence of numbers (which could represent cars or anything else) can converge (mathematically, this time) on a single, specific number. Certain processes, distributions and events can result in convergence— which basically mean the values will get closer and closer together.

The main object of this paper is to study two more extensions of the concept of statistical convergence namely σ-statistical convergence and lacunary σ-statistical convergence. We also study the concept of L_θ-convergence. In section 1.2 we study some inclusion relations between L_θ-convergence and lacunary σ-statistical convergence and show that these are equivalent for bounded sequences. Further in section 1.3 we study relation between σ-statistical convergence and lacunary σ-statistical convergence.

Definition 1.1.1. Let σ be a mapping of the set of positive integers into itself. A continuous linear functional Φ on l_∞, the space of real bounded sequences $x = \{\xi_k\}$, is said to be an invariant mean or a σ-mean if and only if

1. $\Phi(x) \geq 0$ if $\xi_k \geq 0$ for all k,
2. $\Phi(\{\xi_{\sigma(k)}\}) = \Phi(x)$ for all $x \in l_\infty$,
3. $\Phi(e) = 1$ where $e = \{1,1,1,\ldots\}$.

The mappings σ are one-to-one and such that $\sigma^m(k) \neq k$ for all positive integers k and m, where $\sigma^m(k)$ denotes the m^{th} iterate of the mapping σ at k. Thus Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x) = \lim \xi_k$ for all $x \in c$. In case σ is the translation mapping k→k+1, an invariant mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences [19].

If $x = \{\xi_k\}$, set $Tx = \{T\xi_k\} = \{\xi_{\sigma(k)}\}$. It can be shown [28] that

$$V_\sigma = \{x = \{\xi_k\} : \lim_{m\to\infty} t_{mk}(x) = \xi e \text{ uniformly in } k, \xi = \sigma\text{-}\lim \xi_k\}$$

where $t_{mk}(x) = \dfrac{(\xi_k + T\xi_k + \ldots + T^m\xi_k)}{m+1}$.

Several authors including Mursaleen [22], Savas [27], Schaefer [31] and others have studied invariant convergent sequences.

Definition 1.1.2. A sequence $x = \{\xi_k\}$ is said to be strongly σ-convergent [23] to ξ if

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.}$$

In this case we write $\xi_k \to \xi[V_\sigma]$ and $[V_\sigma]$ denotes the set of all strongly σ-convergent sequences.

Remark 1.1.3.

(i) For $\sigma(m) = m+1$, the space $[V_\sigma]$ is the space of strongly almost convergent sequences.

(ii) It is known [23] that $c \subset [V_\sigma] \subset V_\sigma \subset l_\infty$.

Definition 1.1.4. A lacunary sequence is an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$.

Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$.

Definition 1.1.5. Let θ be a lacunary sequence. The space denoted by N_θ is defined [9] as

$$N_\theta = \{x = \{\xi_k\}: \text{for some } \xi, \lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_k - \xi| = 0\}.$$

Definition 1.1.1. A sequence $x = \{\xi_k\}$ is said to be lacunary strong σ-convergent [28] to ξ if

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.}$$

We shall denote by L_θ the set of all lacunary strong σ-convergent sequences.

Remark 1.1.1. $L_\theta \Leftrightarrow [V_\sigma]$ for every lacunary sequence θ.

Definition 1.1.8. A complex number sequence $x = \{\xi_k\}$ is said to be σ-statistically convergent or S_σ -convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{n\to\infty} \frac{1}{n} |\{0 \le k \le n: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0 \text{uniformly in m.}$$

In this case we write S_σ-lim $\xi_k = \xi$ or $\xi_k \to \xi(S_\sigma)$ and S_σ denotes the set of all σ-statistically convergent sequences.

Definition 1.1.9. Let $\theta = \{k_r\}$ be a lacunary sequence. The complex number sequence $x = \{\xi_k\}$ is said to be lacunary σ-statistically convergent or $S_{\sigma\theta}$-convergent to the number ξ if for each $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0 \text{uniformly in m.}$$

In this case we write $S_{\sigma\theta}$-lim $\xi_k = \xi$ or $\xi_k \to \xi(S_{\sigma\theta})$ and $S_{\sigma\theta}$ denotes the set of all lacunary σ-statistically convergent sequences.

1.2 Some Inclusion Relations Between L_Θ-Convergence And Lacunary σ-Statistical Convergence

In his section we study some inclusion relations between L_θ-convergence and lacunary σ-statistical convergence and show that these are equivalent for bounded sequences.

Theorem 1.4.1. Let $\theta = \{k_r\}$ be a lacunary sequence. Then

(i) $\xi_k \to \xi(L_\theta) \Rightarrow \xi_k \to \xi(S_{\sigma\theta})$,

(ii) if $x \in l_\infty$ and $\xi_k \to \xi(S_{\sigma\theta})$, then $\xi_k \to \xi(L_\theta)$,

(iii) $S_{\sigma\theta} \cap l_\infty = L_\theta$.

Proof. (i). Since $\xi_k \to \xi(L_\theta)$, for each $\varepsilon > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = 0$$

uniformly in m. …(1)

If $\varepsilon > 0$, we can write

$$\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| \ge \sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon}} |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

Consequently,

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}|$$

Hence by (1) and the fact that ε is fixed number, we have

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0$$

uniformly in m,

i.e. $\xi_k \to \xi(S_{\sigma\theta})$.

(ii). Suppose that $\xi_k \to \xi(S_{\sigma\theta})$ and $x \in l_\infty$. Then for each $\varepsilon > 0$

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| = 0$$

uniformly in m. ... (2)

Since $x \in l_\infty$, there exists a positive real number M such that $|\xi_{\sigma^k(m)} - \xi| \le M$ for all k and m.

For given $\varepsilon > 0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon}} |\xi_{\sigma^k(m)} - \xi| + \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)} - \xi| < \varepsilon}} |\xi_{\sigma^k(m)} - \xi|$$

$$\le \frac{1}{h_r}\sum_{\substack{k\in I_r \\ |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon}} M + \frac{1}{h_r}\sum_{k\in I_r} \varepsilon = \frac{M}{h_r} |\{k \in I_r: |\xi_{\sigma^k(m)} - \xi| \ge \varepsilon\}| + \varepsilon \frac{1}{h_r} [n-(n-h_r+1) + 1]$$

$$= \frac{M}{h_r} |\{k \in I_r : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon \frac{1}{h_r} h_r$$

$$= \frac{M}{h_r} |\{k \in I_r : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon$$

$$\Rightarrow \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| \leq M \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| + \varepsilon$$

Hence by using (2), we get

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |\xi_{\sigma^k(m)} - \xi| = 0 \quad \text{uniformly in m.} \qquad \ldots(3)$$

$$\Rightarrow \quad \xi_k \to \xi(L_\theta).$$

Example 1.2.2. Let θ be given and define ξ_k to be $1,2,3,\ldots,[\sqrt{h_r}]$ for $k = \sigma^n(m)$, $n = k_{r-1}+1, k_{r-1}+2,\ldots,k_{r-1}+[\sqrt{h_r}]$; $m \geq 1$ and $\xi_k = 0$ otherwise (where [ ] denotes the greatest integer function).

Note that x is not bounded. Now

$$\frac{1}{h_r} |\{k \in I_r : |\xi_{\sigma^k(m)} - 0| \geq \varepsilon\}| = \frac{[\sqrt{h_r}]}{h_r} \to 0 \text{ as } r \to \infty,$$

i.e. $\xi_k \to 0(S_{\sigma\theta})$. But

$$\frac{1}{h_r} \sum_{k\in I_r} |\xi_{\sigma^k(m)} - 0| = \frac{1}{h_r}\left([\sqrt{h_r}]\frac{([\sqrt{h_r}]+1)}{2}\right) \to \frac{1}{2} \neq 0 \text{ as } r \to \infty,$$

i.e. ξ_k □$0(L_\theta)$.

Thus inclusion in (i) is proper and this example shows that the boundedness condition can not be omitted from (ii).

(iii). It follows from (i), (ii), Remark 1.1.7 and the fact that $[V_\sigma] \subset l_\infty$.

This completes the proof of the theorem.

1.3 In this section we study relation between S_σ-convergence and $S_{\sigma\theta}$-convergence. First we discuss a lemma which will be used in studying that relation.

Lemma 1.4.1. A sequence $x = \{\xi_k\}$ is σ-statistically convergent to the number ξ if for given $\varepsilon_1 > 0$ and each $\varepsilon > 0$, there exist n_0 and m_0 such that

$$\frac{1}{n} |\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| < \varepsilon_1$$

for all $n \geq n_0$ and $m \geq m_0$.

Proof. Let $\varepsilon_1 > 0$ be given. For each $\varepsilon > 0$, choose n_0' and m_0 such that

$$\frac{1}{n} |\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| < \frac{\varepsilon_1}{2} \qquad \ldots(4)$$

for all $n \geq n_0'$ and $m \geq m_0$.

It is enough to prove that there exists n_0'' such that for $n \geq n_0''$, $0 \leq m \leq m_0$,

$$\frac{1}{n} |\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| < \varepsilon_1 \qquad \ldots(5)$$

since taking $n_0 = \max\{n_0', n_0''\}$, (5) will hold for $n \geq n_0$ and for all m, which gives the result.

Once m_0 has been chosen, $0 \leq m \leq m_0$, m_0 is fixed.

So let $|\{0 \leq k \leq m_0-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| = K$.

Now taking $0 \leq m \leq m_0$ and $n \geq m_0$, we have

$$\frac{1}{n} |\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| = \frac{1}{n} |\{0 \leq k \leq m_0-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}|$$

$$+ \frac{1}{n} |\{m_0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}|$$

$\leq \frac{1}{n}K + \frac{\varepsilon_1}{2}$ [Using (4)]

$< \varepsilon_1$ [Taking n sufficiently large]

which gives (5), and hence the result follows.

Theorem 1.3.2. $S_{\sigma\theta} = S_\sigma$ for every lacunary sequence θ.

Proof. Let $x \in S_{\sigma\theta}$. Then from Definition 1.1.9, given $\varepsilon_1 > 0$, there exist r_0 and ξ such that

$$\frac{1}{h_r}|\{0 \leq k \leq h_r - 1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| < \varepsilon_1$$

for $r \geq r_0$ and $m = k_{r-1} + 1 + u$, $u \geq 0$.

Let $n \geq h_r$ and write $n = ih_r + t$ where $0 \leq t \leq h_r$, i is an integer. Since $n \geq h_r$, it follows that $i \geq 1$.

Now

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| \leq \frac{1}{n}|\{0 \leq k \leq (i+1)h_r - 1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}|$$

$$= \frac{1}{n}\sum_{j=0}^{i}|\{jh_r \leq k \leq (j+1)h_r - 1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}|$$

$$\leq \frac{1}{n}(i+1)h_r\,\varepsilon_1$$

$$\leq 2i\,h_r\frac{\varepsilon_1}{n} \qquad [i \geq 1]$$

for $\frac{h_r}{n} \leq 1$, since $\frac{ih_r}{n} \leq 1$. So

$$\frac{1}{n}|\{0 \leq k \leq n-1 : |\xi_{\sigma^k(m)} - \xi| \geq \varepsilon\}| \leq 2\varepsilon_1.$$

Then, by Lemma 1.4.1, $x \in S_\sigma$.

Thus $S_{\sigma\theta} \subset S_\sigma$.

It is easy to see that $S_\sigma \subset S_{\sigma\theta}$.

Hence $S_{\sigma\theta} = S_\sigma$ for every lacunary sequence θ.

This completes the proof of the theorem.

Remark 1.3.3. When $\sigma(m) = m + 1$, from Definition 1.1.8 and Definition 1.1.9, we have the definitions of almost statistical convergence and lacunary almost statistical convergence of a sequence.

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