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Crank-Nicolson Analysis of Black-Scholes Partial Differential Equation for Stock Market Prices

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Abstract: This paper presents a partial differential equation that determines Put option prices using both analytic formula and Crank-Nicolson numerical solutions for different stock prices. The deviation values were derived from Black-Scholes analytical solutions, placing certain criteria using three standard statistical tools as a measure for pricing effects. Results obtained revealed when options are overpriced, underpriced as well as no-mispricing which is in line with hypothetical predictions and important improvement over earlier efforts. In the same setting, Kolmogorov-Smirnov (KS) was used to verify the distributions of Black-Scholes (BS) and Crank-Nicolson (CN). In addition, the initial stock prices of no-mispricing were compared with others and results obtained indicated that initial stock price of forty (40) were the most significant and excellent stock price for Put options. This paper presented here has deep connotation for future studies of Put options. Consequently, our novel research contribution is unique and is strongly recommended for use in this area of mathematical finance.
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Key words: Crank-Nicolson, Put Option, Mispricing, Pricing effects, Bias and Modeling.

1. Introduction

This paper considered the Black-Scholes (BS) partial differential equation (PDE) and some of its dynamics of financial markets. BS PDE of option pricing is a key factor in financial Mathematics since it explores the changes of options value as a function of securities and time dependent. The relevance of options valuation was first demonstrated by (Black and Scholes, 1973) when there was difficulty to option traders to value an option at expiration. After the early discovery, a great number of empirical studies have examined the validity of the model such as (Hull, 2003), (Macbeth and Merville, 1979), (Nwobe *et al.,* 2019), (Razali, 2006) and (Rinalini, 2006) and discovered that the model misprices options which will not in any way benefit the option traders.

That is to say the correctness of the model is still questionable and the major part of the model is how to predict the future volatility of the underlying asset (Wokoma and Amadi, 2019), hence determine a correct option price. In this paper, we shall be interested in Crank-Nicolson (CN) finite difference method for valuation of European put option which have gained the interest of researchers for finding approximate solutions to PDEs and this interest is driven by demand of applications of societal problems. From the results of BS model, we develop a new method of assessing pricing effects which will reduce pricing bias of BS model. To this end, the bias in valuation can cause unreasonable loss for traders; valuation is a key feature of trading system.

2. Methodology

The Black-Scholes model is based on seven assumptions as indicated below:

The asset price follows a Brownian motion with μ and σ as constants.

There are no transaction costs or taxes. All securities are perfectly divisible.

There is no dividend during the life of the derivatives.

There are no riskless arbitrage opportunities.

The security trading is continuous.

They gave the formula for the prices of European put option as:

$$\left.\begin{aligned}P &= SN(d_1) - Ke^{-rt}N(d_2)\\ &\frac{in\left(\frac{S}{K}\right)+\left(\frac{r+\sigma^2}{2}\right)T}{\sigma\sqrt{T}}\\ d_2 &= d_1 - \sigma\sqrt{T}\end{aligned}\right\} \quad (1)$$

Where,

P is Price of a put option,

S is price of underlying asset,

K is the strike price,

r is the riskless rate,

T is time to maturity,

σ^2 is variance of underlying asset,

σ is standard deviation of the (generally referred to as volatility) underlying asset, and N is the cumulative normal distribution.

2.1 Derivation of Black-Scholes (BS) Partial Differential Equation (PDE)

According to [9], the derivation of BS PDE is based on Ito process with an assumption that the stock prices follow a geometric Brownian motion, i.e.,

$$dS = \mu Sdt + \sigma Sdx \quad (2)$$

Where,

S is the stock price,

μ is the drift,

σ is the volatility of underlying asset and

dx is a wiener process.

Suppose we have an option whose $V(S,t)$ depends only on S and t. Assuming also that the asset price is perturbed by a small change dS, then the function V will also change. Using Ito's lemma.

$$dV = \sigma S\frac{\partial V}{\partial S}dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t}\right)dt \quad (3)$$

According to [10], the value of one portfolio having one stock can be expressed with the function $V(S,t)$.

$$\pi = V - \Delta S \quad (4)$$

The change in the portfolio at time dt in (4) is given by

$$d\pi = dV - \Delta dS \quad (5)$$

Putting (2), (3) into (5), we find that π follows random walk, given by.

$$d\pi = \sigma S\left(\frac{\partial V}{\partial S} - \Delta\right)dx + \left(\mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu\Delta S\right)dt$$

To eliminate the random component in this random walk, let

$$\Delta = \frac{\partial V}{\partial S} \quad (6)$$

Note that Δ is the value of $\frac{\partial V}{\partial S}$ at the start of the time step dt. This results in a portfolio whose increment is wholly deterministic so that

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt \quad (7)$$

Now that the portfolio is riskless it should earn riskless return. The change in the portfolio at time dt becomes (after substituting (2) and (3) into (5) and dividing through by dt)

$$d\pi = r\pi dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$, and

$$r(V - \Delta S)dt = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}\right)dt$$

This implies that

$$rV - rS\frac{\partial V}{\partial S} = \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}$$

This gives the solution

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0 \quad (8)$$

This is the Black-Scholes partial differential equation.

2.2 European Put Option

The B-S PDE for European put option with value $P(S,t)$ is defined as in (9)

$$\frac{\partial P}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 P}{\partial S^2}+rS\frac{\partial P}{\partial S}-rP=0 \quad (9)$$

With the following initial and boundary conditions:

$$\left.\begin{array}{l} P(0,t)=Ke^{-rt} \\ P(S,t)=0 \ \ when \ \ S\to\infty \\ P(S,T)=\max(K-S,0) \end{array}\right\} \quad (10)$$

2.3 The Crank-Nicolson finite difference Method for an Option pricing Model

The Crank-Nicolson finite difference method is to overcome the stability short-comings by applying the stability and convergence restrictions of the explicit finite difference methods. It is essentially an average of the implicit and explicit methods. However, to carry out a Crank-Nicolson approximation method on Black-Scholes partial differential equation, there will be a price time mesh; the vertical axis in the mesh represents the stock prices, while the horizontal axis represents time.

Thus, each grid point in the mesh denotes a horizontal index i and a vertical index j such that each point in the mesh is the option price for a definite time and a definite stock price. At all times in the mesh $j\Delta s$ is equal to the stock price, and $i\Delta t$ is equal to the time. There exist boundary conditions which aids in the numerical computations; using the pay-off function. The expiration, $t=T$ and the option are computed for all the different stock prices using boundary conditions. To obtain the prices at $t=0$, the model solves backwards for each time step from $t=T$ (Sargon, 2017).

Where,

Recall that the Black-Scholes partial differential equation (8).

Let a function $V(S,t)$ in two dimensional grid points, that is i and j denote the indices for stock price, S and time, t respectively. The function $V(S,t)=V_i^j$; this can be stated with the following difference scheme by Hull (2003).

$$Z_i^j=\frac{1}{2}\sigma^2 S^2 DSS+rS_i DS-rV_i^j \quad (11)$$

Where,

$S=i\Delta s$, for $0\le i\le m$, $t=j\Delta t$ for $0\le j\le i$

$$DSS=\frac{V^j_{i+1}-2V^j_i+V^j_{i-1}}{\Delta^2} \quad (12)$$

$$DS=\frac{V^j_{i+1}-V^j_{i-1}}{2\Delta S} \quad (13)$$

Taking forward difference and backward difference approximations respectively yields implicit and explicit schemes given below.

If we use a forward difference approximation to the time partial derivative we obtain explicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^j=0 \quad (14)$$

And similarly we obtain the implicit scheme

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+Z_i^{j+1}=0 \quad (15)$$

The averages of equations (14) and (15) yields Crank-Nicolson method of approximation.

$$\frac{V_i^{j+1}-V_i^j}{\Delta t}+\frac{1}{2}\left(Z_i^j+Z_j^{j+1}\right)=0 \quad (16)$$

From equation (16)

$$V_i^j-\frac{\Delta t}{2}Z_i^j=V_i^{j+1}+\frac{\Delta t}{2}Z_i^{j+1} \quad (17)$$

$$Z_i^j = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^j - 2V_i^j + V_{i-1}^j\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^j - V_{i-1}^j\right] - rV_i^j$$

and

$$Z_i^{j+1} = \frac{\sigma^2 S_i^2}{2(\Delta S)^2}\left[V_{i+1}^{j+1} - 2V_i^{j+1} + V_{i-1}^{j+1}\right] + \frac{rS_i}{2\Delta S}\left[V_{i+1}^{j+1} - V_{i-1}^{j+1}\right] - rV_i^{j+1}$$

Collecting like terms of $V_{i-1}, V_i \text{ and } V_{i+1}$ and simplifying gives

$$Z_i^j = V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^j\left[\frac{2\Delta t}{\Delta t} + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right] \quad (18)$$

and

$$Z_i^{j+1} = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t S_i}{2\Delta S}\right] + V_i^{j+1}\left[\frac{2\Delta t}{\Delta t} - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t S_i}{2\Delta S}\right] \quad (19)$$

Using (18) and (19) solving simultaneously and taking the average of these two equations we obtain;

$$\left.\begin{array}{l} V_{i-1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta t S_i}{4\Delta S}\right] + V_i^j\left[1 + \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} + \frac{r\Delta t}{2}\right] - V_{i+1}^j\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta t S_i}{4\Delta S}\right] \\ = V_{i-1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} - \frac{r\Delta t S_i}{4\Delta S}\right] + V_i^{j+1}\left[1 - \frac{\Delta t\sigma^2 S_i^2}{2(\Delta S)^2} - \frac{r\Delta t}{2}\right] + V_{i+1}^{j+1}\left[\frac{\Delta t\sigma^2 S_i^2}{4(\Delta S)^2} + \frac{r\Delta t S_i}{4\Delta S}\right] \end{array}\right\} \quad (20)$$

The expressions inside the square brackets of (18) are replaced with the a_i, b_i, c_i. We obtain the following equation

$$a_i V_{i-1}^j + (1+b_i)V_i^j - c_i V_{i+1}^j = a_i V_{i-1}^{j+1} + (1-b_i)V_i^{j+1} + c_i V_{i+1}^{j+1} \quad (21)$$

Where,

$$a_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 - rS_i\right), b_i = -\frac{\Delta t}{2}\left(\sigma^2 S_i^2 - r\right) \text{ and } c_i = \frac{\Delta t}{4}\left(\sigma^2 S_i^2 + rS_i\right).$$

a_i, b_i, c_i are random variables; $i = 0,1,\ldots,M$.

2.4 Modeling Pricing Effects of put options

Let $X_i, i = 1,\ldots,T$, be the Black-Scholes exact values for T trading days and $\bar{X} = \frac{1}{T}\sum X_i$ be the mean of the option value.

Define $d_i = X_i - \bar{X}$, as the deviation from the mean. Define also

$$d_i^{\pm} = \begin{cases} X_i > \bar{X},\ i = 1,2,\ldots,m_1 (\mathrm{m}_1\ \textit{is number of positive}\ d_i) \\ X_i < \bar{X},\ i = 1,2,\ldots,m_2 (m_2\ \textit{is number of negative}\ d_i) \end{cases}$$

$$\textit{The mean of } d_i^+ \textit{ is } \overline{d_i}^+ = \frac{1}{m_1}\sum_{i=1}^{m_1} d_i^+, \textit{ the mean of } d_i^- \textit{ is}$$

$$\overline{d_i}^- = \frac{1}{m_2}\sum_{i=1}^{m_2} d_i^-, \quad \overline{d_i} = \frac{1}{T}\sum_{i=1}^{T} d_i$$

Here, we use standard deviation in creating strategies for investing and trading because it helps measure market volatility and predict performance trends. Note that

$m_1 + m_2 = T$

if and only if $d_i \neq 0,$ otherwise $m_0 + m_1 + m_2 = T$, where

m_0 is the number of zero differences $(d_i = 0).$ The standard deviations of the different components d_i^-, d_i^+ and $d_i, \overline{d}^- < \overline{d} < \overline{d}^+$ are

$$S_{d^+} = \sqrt{\frac{\sum (d_i^+)^2}{m_1 - 1}},\ S_{d^-} = \sqrt{\frac{\sum (d_i^-)^2}{m_2 - 1}},\ S_d = \sqrt{\frac{\sum (d_i)^2}{T - 1}} \qquad (22)$$

For simplicity and without loss of generality, we also define the following:

$S_1, S_2, S_3,$ representing S_{d^+}, S_{d^-} and S_d respectively and

$$\beta_k = \frac{S_k}{\sqrt{\tau}}, k = 1, 2, 3 \qquad (23)$$

Where τ is $\sqrt{252}$ trading days.

2.5 Criteria for selection

Let the index of price function be $\beta_k X_i$,

Where β_k is a constant and X_i a vector. $\beta_k X_i = \{A, B, C\}$ Where $A, B,$ and C are the products of X_i and β_1, β_2 and β_k respectively. The maximum value r_1 will be referred to as overpricing if r_1 =max $\{A, B, C\}$, is middle value, r_2 as under pricing and r_3 as no mispricing if r_3 = min $\{A, B, C\}$ in line with Ekakaa *et al.* (2016) and Etuk (2014).

3. Results and Discussion

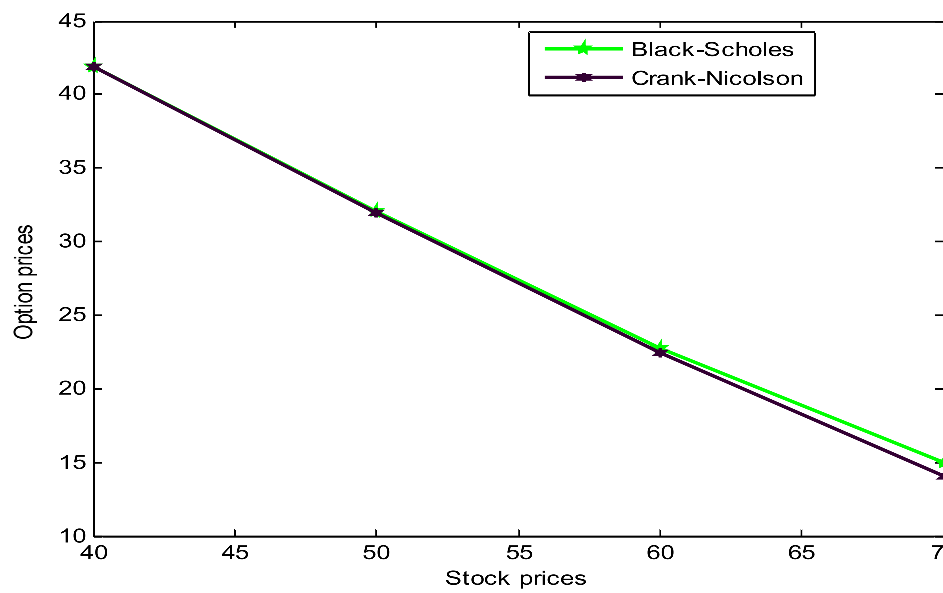

Figure 1: Comparing Black-Scholes with Crank-Nicolson numerical solution under different stock prices for put option.

Values in Tables 1 and 2 were generated by fixing r=0.2, k=100 while allowing S_0 and σ vary in (1) for put options such that $S_0 = (40, 50, 60, 70)$, $\sigma = (0.25, 0.30, ..., 1.00)$ presented in column 1 are the difference values of σ, columns 2 and 3 are the exact values of BS and that of CN respectively. The 4th column gives the relative error (RE) of the difference between the estimated prices using BS and CN pricing schemes given by $|BS - CN| / BS$. RE is the ratio of absolute difference between BS and CN to BS such that when this ratio is very small, the performances of both BS and CN are equivalent, otherwise they are noticeable difference as can be observed in Table 1.

The difference in performance increases from near zero with increasing value of σ. Visual inspection of Figure 2 show that BS and CN are indistinguishable when $0 < \sigma < 0.5$ and the differences start to when σ is in the range $0.5 \leq \sigma \leq 1$. Using same values of σ, r and K as in left panel of Table 1, only S_0 changed from 40 to 50 in the Table1(right panel) panels respectively. From Tables 1 and 2, we observe that besides influence of σ on RE, the initial price S_0 increases both BS exact values and CN approximate values and by implication, increases the RE as between BS and CN with varying values of S_0 and σ.

Table 1: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option with K = 100, r = 0.2 and T = 1

Sigma	$S_o = 40, K = 100$			$S_o = 50, K = 100,$		
	B-S Exact values	C-N	Relative Error	B-S Exact Values	C-N	Relative Error
0.25	41.8817	41.8778	9.3119E-05	32.0183	31.9605	1.8052E-03
0.3	41.9211	41.8977	5.5819E-04	32.2717	32.1011	5.2864E-03
0.35	42.0213	41.9458	1.7967E-03	32.6694	32.3076	0.01107
0.4	42.2025	42.0283	4.1277E-03	33.1966	32.5633	0.01908
0.45	42.4721	42.1446	7.7109E-03	33.8322	32.8510	0.029002
0.5	42.8277	42.2898	0.01256	34.5553	33.1562	0.04049
0.55	43.2622	42.4578	0.01859	35.3479	33.4685	0.05317
0.6	43.7659	42.6422	0.02568	36.1951	33.7805	0.06671
0.65	44.3292	42.8374	0.03365	37.0850	34.0870	0.08084
0.7	44.9429	43.0389	0.04236	38.0079	34.3849	0.09532
0.75	45.5988	43.2432	0.05166	38.9559	34.6722	0.10996
0.8	46.2893	43.4474	0.06139	39.9226	34.9476	0.1246
0.85	47.0083	43.6494	0.07145	40.9028	35.2108	0.1392
0.9	47.7501	43.8478	0.08172	41.8921	35.4616	0.1535
0.95	48.5099	44.0415	0.09211	42.8869	35.7001	0.1676
1.0	49.2837	44.2297	0.10255	43.8839	35.9265	0.1813

Table 2: Comparing the performance between the Black-Scholes exact values and Crank-Nicolson finite difference method for European Put Option with K = 100, r = 0.2 and T = 1

Sigma	$S_o = 60, K = 100$			$S_o = 70, K = 100$		
	B-S Exact values	C-N	Relative Error	B-S Exact Values	C-N	Relative Error
0.25	22.7672	22.4676	0.01316	14.9160	14.0823	0.0559
0.3	23.4960	22.8954	0.02556	16.1862	14.8246	0.0841
0.35	24.3677	23.3644	0.04117	17.5121	15.5168	0.1139
0.4	25.3408	23.8438	0.05907	18.8713	16.1517	0.1441
0.45	26.3860	24.3156	0.07847	20.2504	16.7300	0.1738
0.5	27.4826	24.7698	0.0987	21.6405	17.2554	0.2026
0.55	28.6158	25.2013	0.1193	23.0357	17.7326	0.2302
0.6	29.7745	25.6080	0.1399	24.4317	18.1664	0.2564
0.65	30.9508	25.9896	0.1603	25.8252	18.5617	0.2813
0.7	32.1384	26.3466	0.1802	27.2137	18.9225	0.3047
0.75	33.3325	26.6801	0.1996	28.5950	19.2526	0.3267
0.8	34.5291	26.9914	0.2183	29.9675	19.5554	0.3474
0.85	35.7250	27.2822	0.2363	31.3296	19.8338	0.3669
0.9	36.9177	27.5537	0.2536	32.6800	20.0903	0.3852
0.95	38.1049	27.8076	0.2621	34.0175	20.3272	0.4024
1.0	39.2848	28.0450	0.2861	35.3412	20.5466	0.4186

In Figure 3, overpricing yields an upward trend. Then the underpricing falls in between the two plots which is in line with our criteria for selections of pricing effects. No-mispricing lie along sigma axis indicating no pricing error during the trading period.

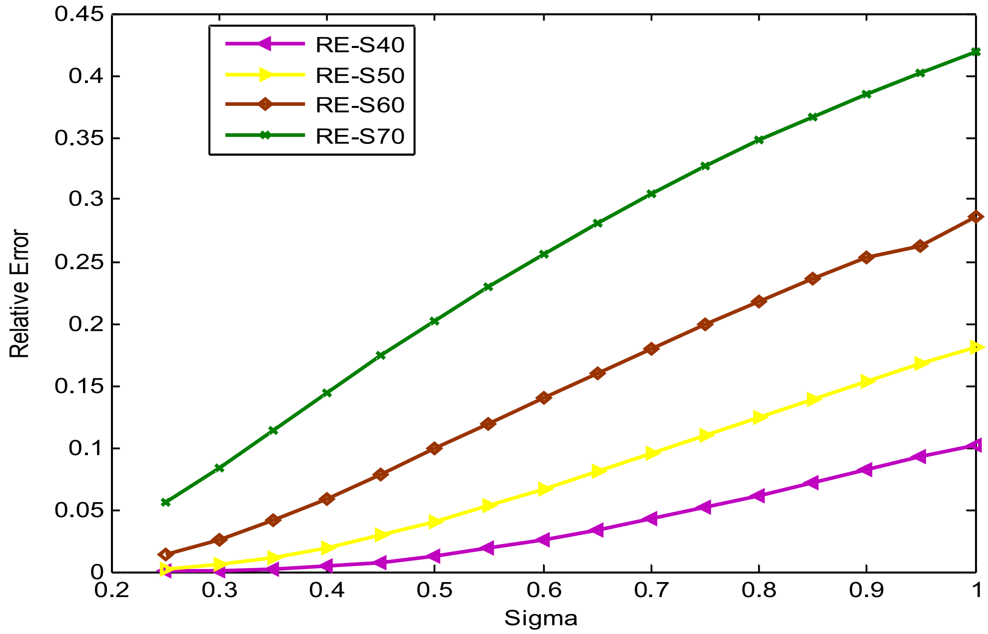

Figure 2: The Levels of relative errors under different initial stock prices for put option.

Table 3: Levels of pricing effects under Put option when initial stock $S_0 = 40$

So = 40, K = 100, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over-pricing	Under-pricing	No Mispricing	$r_1 - r_2 = r_3^*$
0.25	41.8817	41.8778	6.6089	4.1337	2.3119	2.4752
0.3	41.9211	41.8977	6.6151	4.1376	2.3140	2.4775
0.35	42.0213	41.9458	6.6310	4.1475	2.3196	2.4835
0.4	42.2025	42.0283	6.6596	4.1654	2.3296	2.4942
0.45	42.4721	42.1446	6.7021	4.1920	2.3445	2.5101
0.5	42.8277	42.2898	6.7582	4.2271	2.3641	2.5311
0.55	43.2622	42.4578	6.8268	4.2699	2.3881	2.5569
0.6	43.7659	42.6422	6.9063	4.3197	2.4159	2.5866
0.65	44.3292	42.8374	6.9951	4.3753	2.4470	2.6198
0.7	44.9429	43.0389	7.0920	4.4359	2.4808	2.6561
0.75	45.5988	43.2432	7.1955	4.5006	2.5171	2.6949
0.8	46.2893	43.4474	7.3045	4.5688	2.5552	2.7357
0.85	47.0083	43.6494	7.4179	4.6397	2.5949	2.7782
0.9	47.7501	43.8478	7.5350	4.7129	2.6358	2.8221
0.95	48.5099	44.0415	7.6549	4.7879	2.6777	2.867
1.0	49.2837	44.2297	7.7770	4.8643	2.7205	2.9127

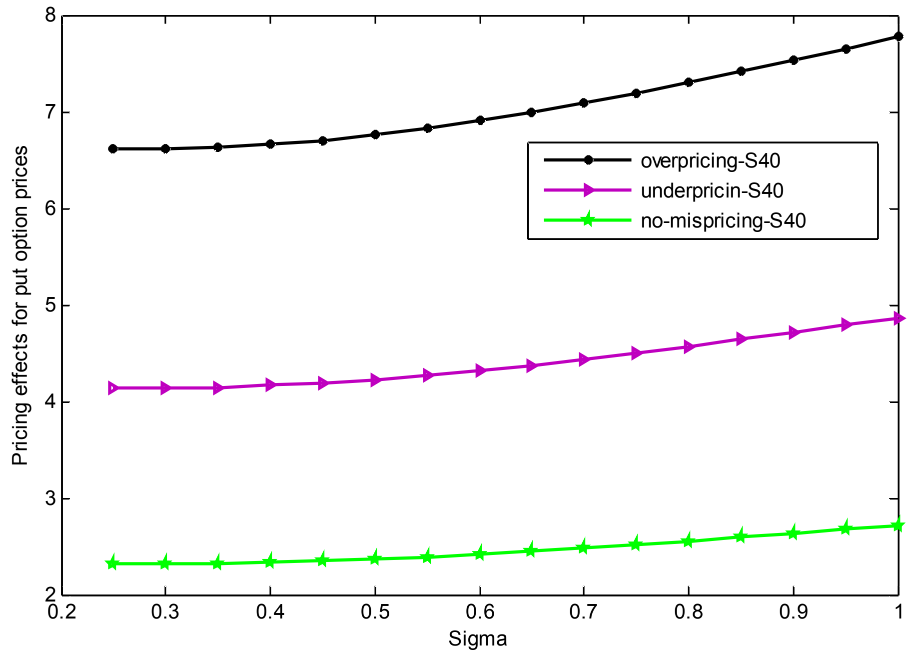

Figure 3: Levels of pricing effects when initial stock is 40 for put option.

Table 4: Levels of pricing effects under Put option when initial stock $S_0 = 50$

So = 50, K = 100, $r = 0.2$						
Sigma	B-S Exact values	C-N	Over –pricing	Under –pricing	No Mis - Pricing	$r_1 - r_2 = r_3^*$
0.25	32.0183	31.9605	7.9694	4.2744	3.6117	3.695
0.3	32.2717	32.1011	8.0324	4.3083	3.6402	3.7241
0.35	32.6694	32.3076	8.1314	4.3614	3.6851	3.77
0.4	33.1966	32.5633	8.2626	4.4317	3.7445	3.8309
0.45	33.8322	32.8510	8.4208	4.5166	3.8163	3.9042
0.5	34.5553	33.1562	8.6008	4.6131	3.8998	3.9877
0.55	35.3479	33.4685	8.7981	4.7189	3.9872	4.0792
0.6	36.1951	33.7805	9.0090	4.8320	4.0828	4.177
0.65	37.0850	34.0870	9.2305	4.9508	4.1832	4.2797
0.7	38.0079	34.3849	9.4602	5.0741	4.2873	4.3861
0.75	38.9559	34.6722	9.6961	5.2006	4.3942	4.4955
0.8	39.9226	34.9476	9.9367	5.3297	4.5033	4.607
0.85	40.9028	35.2108	10.1807	5.4605	4.6138	4.7202
0.9	41.8921	35.4616	10.4269	5.5926	4.7254	4.8343
0.95	42.8869	35.7001	10.6745	5.7254	4.8376	4.9491
1.0	43.8839	35.9265	10.9227	5.8585	4.9501	5.0642

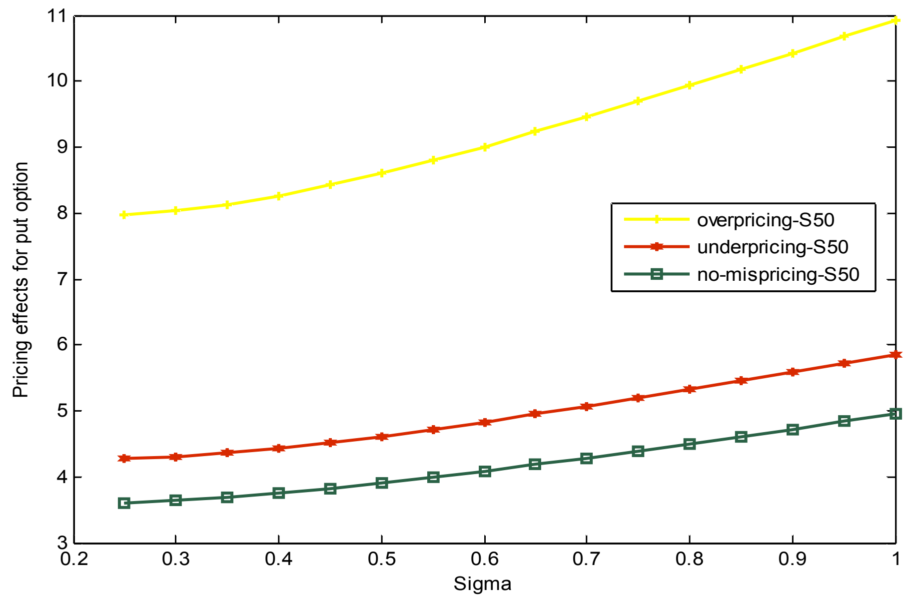

Figure 4: Levels of pricing effects when initial stock is 50 for put option.

It is observed in Figure 4 that the three pricing effects moved upwardly with the same initial stock price of 50. Overpricing plot is an indication of high prices of put options which is on disadvantage to an option trader. Underpricing plots comes very close to no-mispricing which cannot be used for decision making; but for no-mispricing the trend lies along sigma axis but starting from point of origin. Wokoma and Amadi (2019) have reported the contribution of mathematical model in the analysis of the impact of action bitters on the kidney function of Albino Wister Rats with positive outcome.

Table 5: Levels of pricing effects under Put option when initial stock $S_0 = 60$

So = 60, K = 100, $r = 0.2$						
Sigma	BS Exact values	CN	Over-pricing	Under-pricing	No Mis-pricing	$r_1 - r_2 = r_3^*$
0.25	22.7672	22.4676	7.7113	4.1869	3.5653	3.5244
0.3	23.4960	22.8954	7.9581	4.3209	3.6795	3.6372
0.35	24.3677	23.3644	8.2533	4.4812	3.8160	3.7721
0.4	25.3408	23.8438	8.5829	4.6602	3.9684	3.9227
0.45	26.3860	24.3156	8.9369	4.8524	4.1320	4.0845
0.5	27.4826	24.7698	9.3084	5.0541	4.3038	4.2543
0.55	28.6158	25.2013	9.6922	5.2624	4.4812	4.4298
0.6	29.7745	25.6080	10.0846	5.4755	4.6627	4.6091
0.65	30.9508	25.9896	10.4830	5.6919	4.8469	4.7911
0.7	32.1384	26.3466	10.8853	5.9103	5.0329	4.975
0.75	33.3325	26.6801	11.2897	6.1298	5.2199	5.1599
0.8	34.5291	26.9914	11.6950	6.3499	5.4073	5.3451
0.85	35.7250	27.2822	12.1001	6.5698	5.5945	5.5308
0.9	36.9177	27.5537	12.5040	6.7892	5.7813	5.7148
0.95	38.1049	27.8076	12.9061	7.0075	5.9672	5.8986
1.0	39.2848	28.0450	13.3058	7.2245	6.1520	6.0813

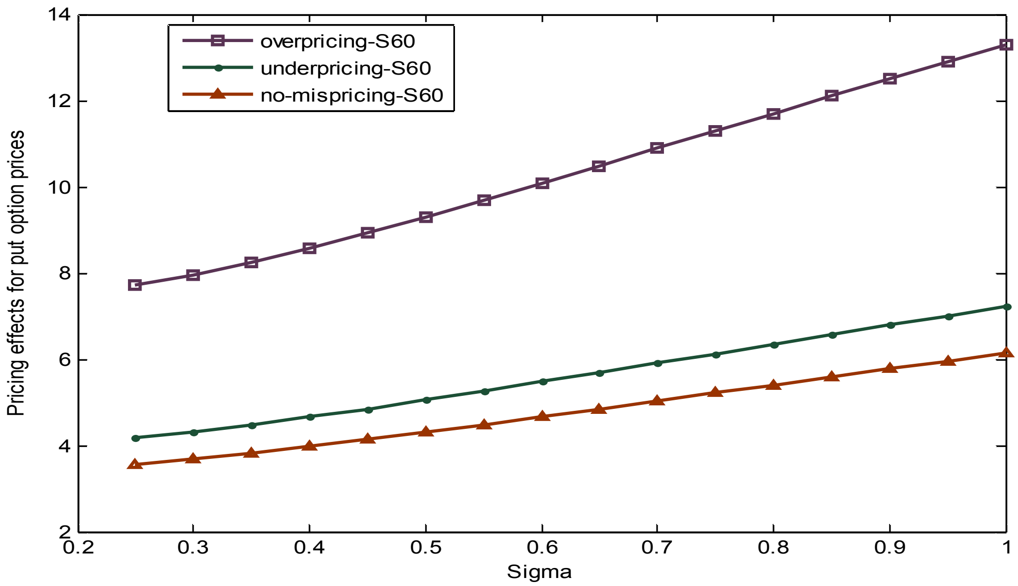

Figure 5: Levels of pricing effects when initial stock is 60 for put option.

As shown in Figure 5, the overpricing effect increased on steady trend. This is true because by definition. It means to charge too high a price for product. The underpricing effect moved below the mean level of the entire plot which is an indication for underpricing of option values. So no-mispricing still falls within the region starting from the origin lied along sigma axis; this is an impression of goods and services of the option trader which matches the intrinsic value of the item during the trading days of one year. In Figure 6 scenario, we observed that overpricing effect showed an upward trend. Underpricing merged with no-mispricing because of price differences. With this two pricing effects profit making issue.

In Figure 7, the plot reveals that the initial stock price of 40 remains the best for put options.

Table 6: Levels of pricing effects under Put option when initial stock $S_0 = 70$

So = 70, K = 100, $r = 0.2$						
Sigma	BS Exact values	CN	Over-pricing	Under-pricing	No Mis-pricing	$r_1 - r_2 = r_3^*$
0.25	14.9160	14.0823	6.1439	3.1413	3.1309	3.013
0.3	16.1862	14.8246	6.6671	3.4088	3.3975	3.2583
0.35	17.5121	15.5168	7.2132	3.6880	3.6758	3.5252
0.4	18.8713	16.1517	7.7731	3.9743	3.9611	3.7988
0.45	20.2504	16.7300	8.3411	4.2647	4.2506	4.0764
0.5	21.6405	17.2554	8.9137	4.5575	4.5423	4.3562
0.55	23.0357	17.7326	9.4884	4.8513	4.8352	4.6371
0.6	24.4317	18.1664	10.0634	5.1453	5.1282	4.9181
0.65	25.8252	18.5617	10.6374	5.4388	5.4207	5.1986
0.7	27.2137	18.9225	11.2093	5.7312	5.7122	5.4781
0.75	28.5950	19.2526	11.7783	6.0221	6.0021	5.7562
0.8	29.9675	19.5554	12.3436	6.3112	6.2902	6.0324
0.85	31.3296	19.8338	12.9047	6.5980	6.5761	6.3067
0.9	32.6800	20.0903	13.4609	6.8824	6.8595	6.5785
0.95	34.0175	20.3272	14.0118	7.1641	7.1403	6.8477
1.0	35.3412	20.5466	14.5570	7.4429	7.4181	7.1141

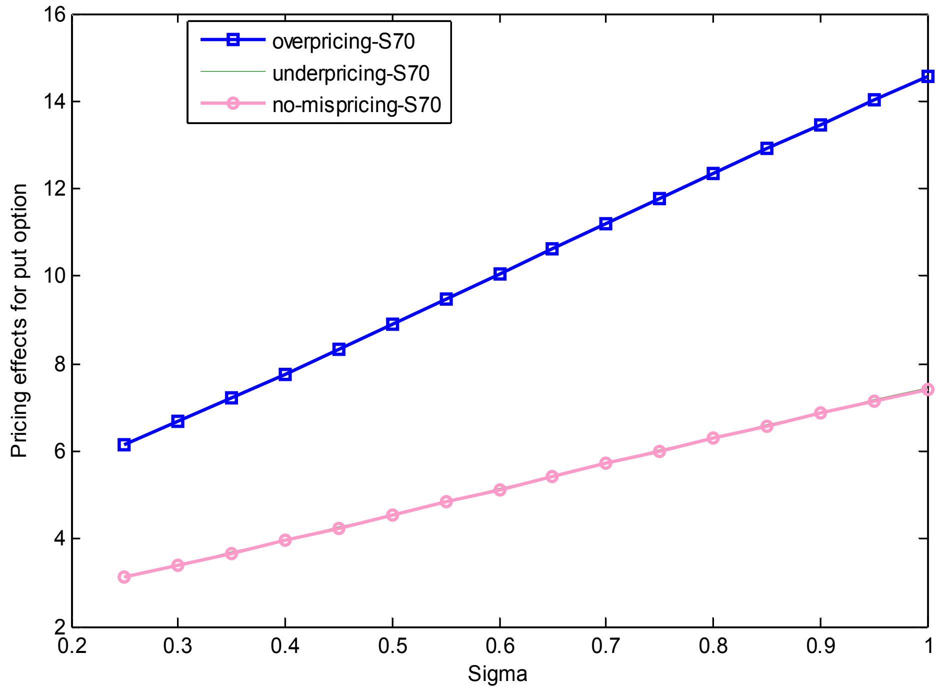

Figure 6: Levels of pricing effects when initial stock is 70 for put option.

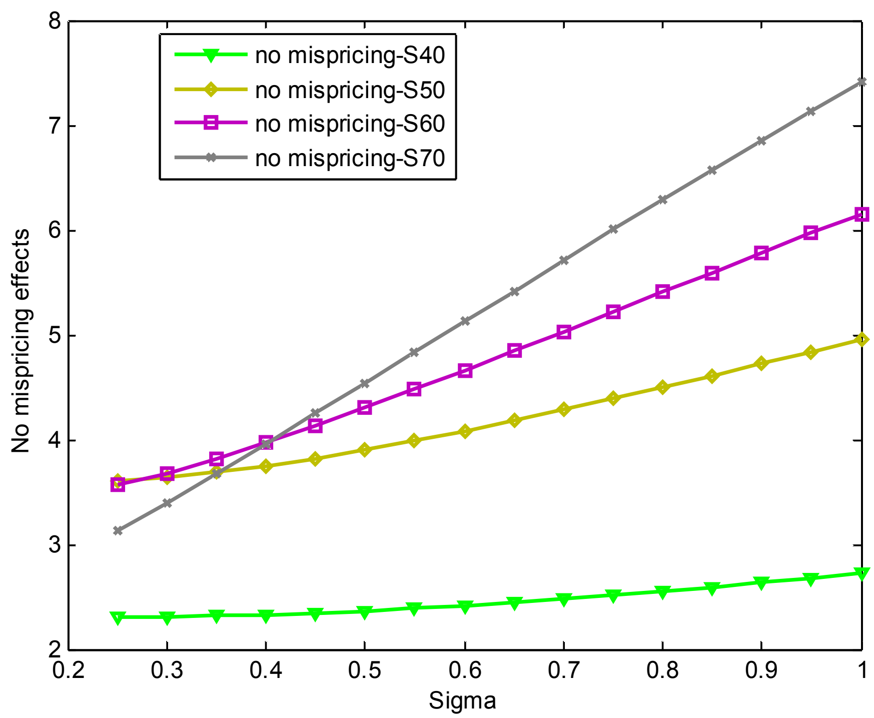

Figure 7: Plots of no-mispricing effects with different initial stock prices.

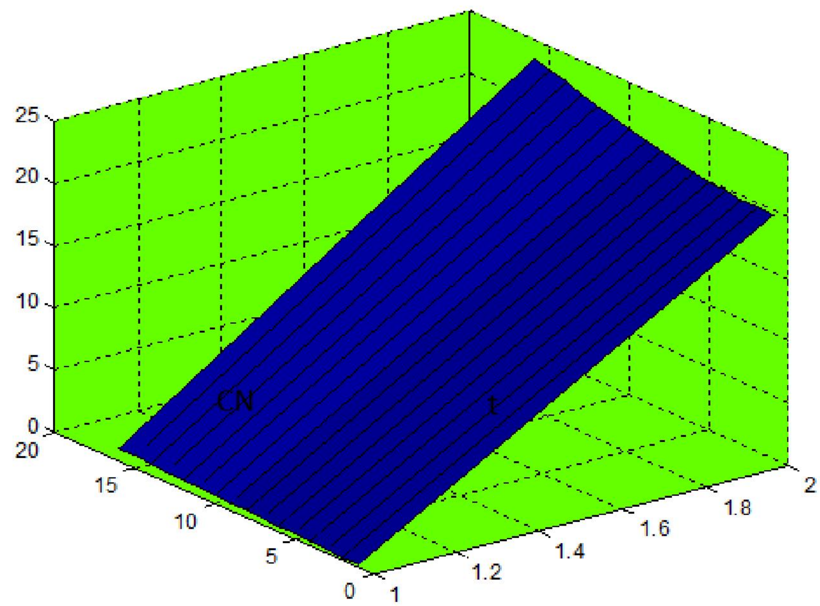

Figure 8: Surface view of BS put option when the initial stock price is 40 with variations of sigma

Figure 8 is skewed to the right; visual inspection shows a constant increase as σ increases. This plot agrees with result of Black and Scholes of 1973 that sigma are constant throughout the life of an option.

H_0 : The BS put option and CN numerical solutions are from the same distribution.

H_1 : They are not from the same distribution.

Table7: Kolmogorov-Simirnov test for BS and CN values come from a common distribution for Put option.

	S_0			
α	40	50	60	70
0.01	Reject	Accept	Accept	Accept
0.05	Accept	Accept	Accept	Accept

In Table 7, we noticed that at $\alpha =$ 0.01 with $S_0 =$ 40, H_0 was rejected but when $S_0 =$ 50, 60 and 70, H_1 was accepted. So when level of significance increased to 0.05 using various initial stock prices of 40, 50, 60 and 70 H_1 was accepted. We can conclude by saying there is significant difference between BS and CN.

4. Conclusion and Recommendation

In this work, the analytical formula for valuing European Put options has been valued as well as Crank-Nicolson numerical solution for different stock prices. However, because of biasedness of BS PDE in mispricing options we developed a new method of assessing pricing effects on the premise to reduce pricing bias. Also all the initial stock prices of "no mispricing" were compared in Figure 8; results showed that initial stock price of 40 are the best for put options.

Thus, put option can only be exercised when the actual price of the asset is less than the strike price of the asset at expiration date. This option is said to be in the money. In another scenario, the model was subjected to goodness of fit test using KS, the test revealed that put options were statistically significant. Consequently, our novel research contribution is unique and is strongly recommended for use in this area of mathematical finance.

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