New Contractive Conditions Of Integral Type And Fixed Point Theorems In Cone Metric Space

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Abstract: The aim of this paper is to extend the concept of F. Khojasteh, Z. Goodarzi and A. Razani to some new contractive conditions of integral type in cone metric space.
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1. Introduction:

The concept of cone metric space was introduced by Huang and Zhang [1] in 2007 and some fixed point theorems was proved. Initially Branciari [2] introduced the contractive condition of integral type and extended Banach fixed point theorem. Later on F. Khojasteh, Z. Goodarzi and A. Razani [3] gave the concept of cone integrable function and proved Branciari's theorem in cone metric space. The aim of this paper is to extend the concept of [3], to some new contractive conditions of integral type in cone metric space.

The following definitions and lemmas are useful for us to prove the main results.

Definition 1.1[1]**:** Let E be a real Banach space and P a subset of E. P is called a cone if the following hold.

(1) P is closed, non-empty and $P \neq \{0\}$.

(2) If $a,\ b \in R$ and $a,\ b \geq 0$, then $ax + by \in P$, $\forall\ x,\ y \in P$.

(3) $x \in P$ and $-x \in P$ implies $x = 0$.

Let $P \subseteq E$ be a cone. We define a partial ordering with respect to P as $x \leq y$ if and only if $y - x \in P$ and $x < y$ will imply that $x \leq y$ but $x \neq y$, while $x << y$ will mean that $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that $0 \leq x < y$ implies $\|x\| \leq M\|y\|$ $\forall\ x,\ y \in E$. The least positive number M is called the normal constant.

Example: Suppose $E = R^2$, $P = \{(x,\ y) \in E \mid x,\ y \geq 0\}$. $X = R$. Let $d : X \times X \to E$ be defined as $d(x,\ y) = (b|x - y|,\ |x - y|)$ where $b \in R$ and $b \geq 0$. Then $(X,\ d)$ is cone metric space.

Definition 1.2[1]**:** Let $(X,\ d)$ be a cone metric space and let $\{x_n\}$ be a sequence in X. Then

(1) $\{x_n\}$ is said to converges to some $x \in X$ if for every $c \in E$ with $0 << c$, $\exists$ a natural number N such that $\forall\ n \geq N$, $d(x_n,\ x) << c$.

(2) $\{x_n\}$ is said to be Cauchy sequence if for every $c \in E$ with $0 << c$, $\exists$ a natural number N such that $\forall\ m,\ n \geq N$, $d(x_n,\ x_m) << c$. $\forall\, x, y \in P$

(3) A cone metric space $(X,\ d)$ is complete if every Cauchy sequence is convergent.

Definition 1.3[3]**:** Let P be a normal cone in E and $\alpha,\ \beta \in E$ where $\alpha < \beta$. Then we define

$$[\alpha,\ \beta] = \{x \in E : s\beta + (1-s)\alpha,\ s \in [0,\ 1]\},$$

$$[\alpha,\ \beta) = \{x \in E : s\beta + (1-s)\alpha,\ s \in [0,\ 1)\}.$$

Definition 1.4[3]: The set $P_1=\{\alpha=x_0,\, x_1,\, x_2,\, \ldots, x_n=\beta\}$ is called a partition of $[\alpha,\beta]$ if and only if the sets $\{[x_{j-1},\, x_j)\}_{j=1}^{n}$ are pairwise disjoint and $[\alpha,\, \beta]=\{\cup_{j=1}^{n}[x_{j-1},\, x_j)\}\cup\{\beta\}$.

Definition 1.5[3]: Let $P_1=\{\alpha=x_0,\, x_1,\, x_2,\, \ldots,\, x_n=\beta\}$ be a partition of $[\alpha,\beta]$ and $\phi=[\alpha,\, \beta]\to P$ be an increasing function. We define cone lower sum and cone upper sum as

$$L_{n^-}^{con}(\phi,\, P_1)=\sum_{j=0}^{n-1}\phi(x_j)\,\|x_j-x_{j+1}\|,$$

$$U_n^{con}(\phi,\, P_1)=\sum_{j=0}^{n-1}\phi(x_{j+1})\,\|x_j-x_{j+1}\|$$, respectively.

The function ϕ is called cone integrable function on $[\alpha,\beta]$ if and only if for all partitions P_1 of $[\alpha,\beta]$

$$\lim_n L_n^{con}(\phi,\, P_1)=S^{con}=\lim_n U_n^{con}(\phi,\, P_1),$$

where S^{con} is unique. We shall write $S^{con}=\int_\alpha^\beta \phi\, dp$ or $\int_\alpha^\beta \phi(t)\, dp(t)$.

Lemma 1.1[3]: If $[\alpha,\beta]\subseteq[\alpha,\, \gamma]$ then $\int_\alpha^\beta \phi\, dp \le \int_\alpha^\gamma \phi\, dp$ for $\phi\in\ell^1(X,\, P)$

$\int_\alpha^\beta (a\phi_1+b\phi_2)dp=a\int_\alpha^\beta \phi_1\, dp+b\int_\alpha^\beta \phi_2\, dp$ for $\phi_1,\, \phi_2\in\ell^1(X,\, P)$ and $a,\, b\in R$

where $\ell^1(X,\, P)$ denotes the set all cone integrable functions.

Definition 1.6[3]: A function $\phi : P\to ⊔$ is said to be subadditive cone integrable function if and only if $\forall\ \alpha,\, \beta\in P$

$$\int_0^{\alpha+\beta} \phi\, dp \le \int_0^\alpha \phi\, dp+\int_0^\beta \phi\, dp.$$

2. Main Results:

Theorem 2.1: Let $(X,\, d)$ be a complete cone metric space with normal cone *P*. Let $\phi : P\to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha,\, \beta]\subset P$ for which $\int_0^\varepsilon \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X\to X$ be a mapping such that

$$\int_0^{d(T(x),T(y))} \phi\, dp \le c\int_0^{d(x,T(y))+d(y,T(x))} \phi\, dp \quad \text{for each } x,\, y\in X,\ c\in\left(0,\, \frac{1}{2}\right).$$

Then *T* has a unique fixed point in *X*.

Proof: Let $x\in X$, choose $x_1\in X$ such that $x_1=T(x)$. Let $x_2\in X$ be such that $x_2=T(x)$. Continuing in this way we can define $x_n=T(x_{n-1})=T^n(x)$ for $n=1,\, 2,\, 3,\, \ldots$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp$$

$$\le c\int_0^{d(x_n,x_n)+d(x_{n-1},x_{n+1})} \phi\, dp$$

$$\le c\int_0^{d(x_{n-1},x_{n+1})} \phi\, dp$$

But $d(x_{n-1},\, x_{n+1})\le d(x_{n-1},x_n)+d(x_n,x_{n+1})$, therefore

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \ \le c\int_0^{d(x_{n-1},x_n)+d(x_n,x_{n+1})} \phi\, dp$$

Since ϕ is cone subadditive, so

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le c\int_0^{d(x_{n-1},x_n)} \phi\, dp + c\int_0^{d(x_n,x_{n+1})} \phi\, dp$$

$$\Rightarrow \int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{c}{1-c}\int_0^{d(x_n,x_{n-1})} \phi\, dp = k\int_0^{d(x_n,x_{n-1})} \phi\, dp, \quad \text{where } k=\frac{c}{1-c}$$

$$\vdots$$

$$\le k^n \int_0^{d(x_1,x_0)} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le k^n \int_0^{d(T(x),x)} \phi\, dp$$

Since $0 \le k < 1$, and $\int_0^{\varepsilon} \phi\, dp >> 0$ for each $\varepsilon >> 0$, so

$$\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0,$$

which implies, that $\lim_n d(x_{n+1},x_n) = 0$.

To show $\{x_n\}$ is Cauchy sequence, we shall show that $\lim_{n\to\infty} d(T(x_{n+\rho}),T(x_n)) = 0$ for each positive integer ρ.

Let $\rho > 0$ be any integer. By triangular inequality

$$d(x_{n+\rho},x_n) \le d(x_{n+\rho},x_{n+\rho-1}) + d(x_{n+\rho-1},x_{n+\rho-2}) + \ldots + d(x_{n+1},x_n)$$

$$\int_0^{d(x_{n+\rho},x_n)} \phi\, dp \le \int_0^{d(x_{n+\rho},x_{n+\rho-1})+\ldots+d(x_{n+1},x_n)} \phi\, dp$$

$$\int_0^{d(T(x_{n+\rho+1}),T(x_n))} \phi\, dp = \int_0^{d(x_{n+\rho},x_n)} \phi\, dp \le \int_0^{d(x_{n+\rho},x_{n+\rho-1})+\ldots+d(x_{n+1},x_n)} \phi\, dp$$

Since ϕ is cone subadditive

$$\le \int_0^{d(x_{n+\rho},x_{n+\rho-1})} \phi\, dp + \int_0^{d(x_{n+\rho-1},x_{n+\rho-2})} \phi\, dp + \ldots + \int_0^{d(x_{n+1},x_1)} \phi\, dp$$

$$\le \left(k^{n+\rho-1} + k^{n+\rho-2} + \ldots + k^n\right)\int_0^{d(x_1,x_0)} \phi\, dp$$

$$\le \left(k^n + k^{n+1} + \ldots + k^{n+\rho-2} + k^{n+\rho-1}\right)\int_0^{d(T(x),x)} \phi\, dp$$

$$\le \frac{k^n}{1-k}\int_0^{d(T(x),x)} \phi\, dp$$

Letting $n \to \infty$, $\lim_{n\to\infty} \int_0^{d(T(x_{n+\rho+1}),T(x_n))} \phi\, dp = 0$.

Which implies that $\lim_{n\to\infty} d(T(x_{n+\rho}),T(x_n)) = 0$ for each positive integer ρ.

Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete cone metric space so $\{x_n\}$ is convergent to some $z \in X$. i.e. $\lim_n x_n = z$.

$$\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$$

$$\le c\int_0^{d(z,x_{n+1})+d(x_n,T(z))} \phi\, dp$$

$$\leq c \int_0^{d(z,x_{n+1})} \phi\, dp + c \int_0^{d(x_n,T(z))} \phi\, dp$$

As $n \to \infty$

$$\int_0^{d(T(z),z)} \phi\, dp \leq c \int_0^{d(z,T(z))} \phi\, dp$$

which implies that $d(T(z),z)=0$ i.e. $T(z)=z$.

Thus z is a fixed point of T.

Uniqueness: Let T has two fixed point z and w i.e. $T(z)=z$ and $T(w)=w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp \leq c \int_0^{d(z,T(w))+d(w,T(z))} \phi\, dp$$

$$\leq c \int_0^{d(z,w)} \phi\, dp + c \int_0^{d(w,z)} \phi\, dp$$

$$\Rightarrow \int_0^{d(z,w)} \phi\, dp \leq \frac{c}{1-c} \int_0^{d(z,w)} \phi\, dp = k \int_0^{d(z,w)} \phi\, dp \quad \text{where } k = \frac{c}{1-c}$$

Which implies that $d(z, w)=0$ i.e. $z=w$.

This shows that T has a unique fixed point in X.

Theorem 2.2: Let (X, d) be a complete cone metric space with normal cone P. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta] \subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$$\int_0^{d(T(x),T(y))} \phi\, dp \leq a \int_0^{d(x,y)} \phi\, dp + b \int_0^{d(y,T(x))} \phi\, dp$$

. For $a, b \in R$ s.t. $a < 1-2b$ and $0 \leq b < \frac{1}{2}$.

Then T has unique fixed point.

Proof: Let $x \in X$, choose $x_1 \in X$ such that $x_1 = T(x)$. Let $x_2 \in X$ be such that $x_2 = T(x)$. Continuing in this way we can define $x_n = T(x_{n-1}) = T^n(x)$ for $n = 1, 2, 3, \ldots$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp$$

$$\leq a \int_0^{d(x_n,x_{n-1})} \phi\, dp + b \int_0^{d(x_{n-1},x_{n+1})} \phi\, dp$$

Using triangle inequality and cone subadditivity,

$$\leq a \int_0^{d(x_n,x_{n-1})} \phi\, dp + b \int_0^{d(x_{n-1},x_n)} \phi\, dp + b \int_0^{d(x_n,x_{n+1})} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \leq \frac{a+b}{1-b} \int_0^{d(x_n,x_{n-1})} \phi\, dp = k \int_0^{d(x_n,x_{n-1})} \phi\, dp \quad \text{, where } k = \frac{a+b}{1-b}$$

$$\vdots$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \leq k^n \int_0^{d(x_1,x_0)} \phi\, dp = k^n \int_0^{d(T(x),x)} \phi\, dp$$

Since $k = \frac{a+b}{1-b} < 1$ then as $n \to \infty$, $\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0$

Which implies that $\lim_n d(x_{n+1}, x_n) = 0$.

It is easy to show that $\{x_n\}$ is a Cauchy sequence (See previous theorem). Since X is complete cone metric space so there is some $z \in X$ such that $\lim_n x_n = z$.

Now, $\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$

$\le a\int_0^{d(z,x_n)} \phi\, dp + b\int_0^{d(x_n,T(z))} \phi\, dp$

As $n\to\infty$, $\int_0^{d(T(z),z)} \phi\, dp \le b\int_0^{d(z,T(z))} \phi\, dp$

Since $0 \le b < \frac{1}{2}$ then $\int_0^{d(T(z),z)} \phi\, dp = 0$ which implies that $d(T(z), z) = 0 \Rightarrow T(z) = z$.

Uniqueness: Let *T* has two fixed point *z* and *w* i.e. $T(z) = z$ and $T(w) = w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$

$\le a\int_0^{d(z,w)} \phi\, dp + b\int_0^{d(w,T(z))} \phi\, dp$

$= (a+b)\int_0^{d(z,w)} \phi\, dp$.

Since $0 < a+b < 1$ therefore

$\int_0^{d(z,w)} \phi\, dp = 0$

$\Rightarrow d(z,w) = 0$

$\Rightarrow z = w$.

It shows that *T* has a unique fixed point.

Theorem 2.3: Let (X, d) be a complete cone metric space with normal cone *P*. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta] \subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$\int_0^{d(T(x),T(y))} \phi\, dp \le c\int_0^{d(x,T(x))+d(y,T(y))} \phi\, dp$. For $c \in \left(0, \frac{1}{2}\right)$ then *T* has a unique fixed point in *X*.

Proof: Let $x \in X$, choose $x_1 \in X$ such that $x_1 = T(x)$. Let $x_2 \in X$ be such that $x_2 = T(x)$. Continuing in this way we can define $x_n = T(x_{n-1}) = T^n(x)$ for $n = 1, 2, 3, ...$

$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp \le c\int_0^{d(x_n,x_{n+1})+d(x_{n-1},x_n)} \phi\, dp$

$\le c\int_0^{d(x_n,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,x_{n-1})} \phi\, dp$

$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{c}{1-c}\int_0^{d(x_n,x_{n-1})} \phi\, dp = k\int_0^{d(x_n,x_{n-1})} \phi\, dp$

As in theorems (2.1), it is easy to prove that $\{x_n\}$ is a Cauchy sequence and completeness of *X* implies that there is some $z \in X$ such that $\lim_n x_n = z$.

Now, $\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$

$\le c\int_0^{d(z,T(z))+d(x_n,x_{n+1})} \phi\, dp$

$\le c\int_0^{d(z,T(z))} \phi\, dp + c\int_0^{d(x_n,x_{n+1})} \phi\, dp$

As $n\to\infty$, $\int_0^{d(T(z),z)} \phi\, dp \le c\int_0^{d(T(z),z)} \phi\, dp$ which implies that $d(T(z), z) \Rightarrow T(z)=z$.

Uniqueness: Let T has two fixed point z and w i.e. $T(z)=z$ and $T(w)=w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$
$$\le c\int_0^{d(z,T(z))+d(w,T(w))} \phi\, dp$$
$$\le c\int_0^{d(z,T(z))} \phi\, dp + c\int_0^{d(w,T(w))} \phi\, dp = 0 \quad \Rightarrow d(z,w)=0 \Rightarrow z=w.$$

Theorem 2.4: Let (X, d) be a complete cone metric space with normal cone P. Let $\phi : P \to P$ be a nonvanishing and subadditive cone integrable map on each $[\alpha, \beta]\subset P$ for which $\int_0^{\varepsilon} \phi\, dp >> 0$, $\varepsilon >> 0$. Let $T : X \to X$ be a mapping such that

$\int_0^{d(T(x),T(y))} \phi\, dp \le c\int_0^{d(x,T(y))+d(y,T(x))+d(x,y)} \phi\, dp$. For some $c\in\left(0,\frac{1}{3}\right)$ than T has a unique fixed point in X.

Proof: Let $x \in X$, define $x_{n+1}=T(x_n)$ for $n\ge 1$ and $x_1=T(x_0)=T(x)$.

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp = \int_0^{d(T(x_n),T(x_{n-1}))} \phi\, dp$$
$$\le c\int_0^{d(x_n,x_n)+d(x_{n-1},x_{n+1})+d(x_n,x_{n-1})} \phi\, dp$$
$$\le c\int_0^{d(x_{n-1},x_{n+1})} \phi\, dp + c\int_0^{d(x_n,x_{n-1})} \phi\, dp$$

Using triangular inequality and cone subadditivity.

$$\le c\int_0^{d(x_{n-1},x_n)} \phi\, dp + c\int_0^{d(x_n,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,x_{n-1})} \phi\, dp$$

$$\int_0^{d(x_{n+1},x_n)} \phi\, dp \le \frac{2c}{1-c}\int_0^{d(x_n,x_{n-1})} \phi\, dp$$
$$\vdots$$
$$\le \left(\frac{2c}{1-c}\right)^n \int_0^{d(x_1,x_0)} \phi\, dp = \left(\frac{2c}{1-c}\right)^n \int_0^{d(T(x),x)} \phi\, dp$$

If $0<\frac{2c}{1-c}<1$ i.e. $c<\frac{1}{3}$ then

$$\lim_n \int_0^{d(x_{n+1},x_n)} \phi\, dp = 0,$$

which implies that $\lim_n d(x_{n+1}, x_n)=0$.

It is easy to prove that $\{x_n\}$ is Cauchy sequence. Since X is complete cone metric space so there is some $z\in X$ such that $\lim_n x_n = z$.

Now,
$$\int_0^{d(T(z),x_{n+1})} \phi\, dp = \int_0^{d(T(z),T(x_n))} \phi\, dp$$
$$\le c\int_0^{d(z,x_{n+1})+d(x_n,T(z))+d(z,x_n)} \phi\, dp$$

$$\leq c\int_0^{d(z,x_{n+1})} \phi\, dp + c\int_0^{d(x_n,T(z))} \phi\, dp + c\int_0^{d(z,x_n)} \phi\, dp$$

As $n \to \infty$, $\int_0^{d(T(z),z)} \phi\, dp \leq c\int_0^{d(z,T(z))} \phi\, dp$

Which implies that $d(T(z),z)=0$. i.e. $T(z)=z$.
Hence z is a fixed point of T.

Uniqueness: Let z and w are two fixed points of T. i.e. $T(z)=z$ and $T(w)=w$.

$$\int_0^{d(z,w)} \phi\, dp = \int_0^{d(T(z),T(w))} \phi\, dp$$

$$\leq c\int_0^{d(z,T(w))+d(w,T(z))+d(z,w)} \phi\, dp$$

$$\int_0^{d(z,w)} \phi\, dp \leq c\int_0^{3d(z,w)} \phi\, dp.$$

Which is possible if $d(z,w)=0$ i.e. $z=w$.
Thus fixed point of T is unique.

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