A Useful technique for solving the differential equation with boundary values

Dr. Dinesh Verma

Associate Professor Department of Applied Science
Yogananda College of Engineering and Technology (YCET), Jammu (J & K)

Abstract: The Laplace transformation is applied in different areas of science, engineering and technology. The Laplace transformation is applicable in so many fields. Laplace transformation is used in solving the time domain function by converting it into frequency domain. Here we have applied Laplace transformation in linear ordinary differential equations with constant coefficient and several ordinary equations wherein the coefficients are variable. Laplace transformation makes it easier to solve the problems in engineering applications and makes differential equations simple to solve. This paper presents a new technological approach to solve Ordinary differential equation with constant coefficient. Some of the important properties are detailed deeply in this paper with proof, brief description of inverse Laplace transformation with effective examples and solve the differential equations with boundary values.

[Dinesh Verma. **A Useful technique for solving the differential equation with boundary values.** *Academ Arena* 2019;11(2):77-79]. ISSN 1553-992X (print); ISSN 2158-771X (online). http://www.sciencepub.net/academia. 6. doi:10.7537/marsaaj110219.06.

Key words: Laplace transformation, Inverse Laplace transformation, Ordinary differential equation (ODE).

Sub area: Laplace transformation
Broad area: Mathematics

Introduction

Laplace transformation is very useful tool in various areas of engineering and science. It helps us to find the solution of initial value problems involving homogeneous and non- homogeneous equations, it minimizes the problem of solving differential equations to an algebraic problem which becomes much easier to solve. In this paper we look that when Laplace transformation is applied to a differential equation, it would change derivatives into algebraic expression in term of p.

It is very powerful technique, because it replaces operations of calculus by operation of algebra i.e. with the application s of Laplace transformation to an initial value problem consisting of an ordinary differential equation together with initial condition is reduced to a problem of solving an algebraic equation.

Definition

Let f (t) is a well defined function of t for all t ≥ 0. The Laplace transformation of f (t), denoted by f (p) or L {F (t)}, is defined as

L {F (t)} $=\int_0^{\infty} e^{-pt} F(t)dt$

Provided that the integral exists, i.e. convergent. If the integral is convergent for some value of p, then the Laplace transformation of F (t) exists otherwise not. Where p is the parameter which may be real or complex number and L is the Laplace transformation operator.

The Laplace transformation of F (t) i.e. $\int_0^{\infty} e^{-pt} F(t)dt$ exists for p>a, if

F (t) is continuous and $\lim_{n\to\infty}\{e^{-at} F(t)\}$ is finite. It should however, be remember that above condition are sufficient and not necessary.

Laplace transformation of elementary function:

1. $L\{1\} = \frac{1}{p}, p > 0$
2. $L\{t^n\} = \frac{n!}{p^{n+1}}, where\ n = 0,1,2,3 \ldots\ldots\ldots.$
3. $L\{e^{at}\} = \frac{1}{p-a},, p > a$
4. $L\{sinat\} = \frac{a}{p^2+a^2}, p > a$
5. $L\{sinhat\} = \frac{a}{p^2-a^2}, p > |a|$
6. $L\{cosat\} = \frac{p}{p^2+a^2}, p > 0$
7. $L\{coshat\} = \frac{p}{p^2-a^2}, p > |a|$

Proof: By the definition of Laplace transformation, we know that L {F (t)} $=\int_0^{\infty} e^{-pt} F(t)d$ then 1. L { e^{at} } $=\int_0^{\infty} e^{-pt}\, 1dt$

$= -\frac{1}{p-a}\left(e^{-\infty} - e^{-0} \right) = \frac{1}{p-a}(0-1)$

$$= \frac{1}{p-a} = f(p), p > a$$

. L {F (t)} $=\int_0^{\infty} e^{-pt} F(t)dt$ then.

2. L { $sinhat$ } $=\int_0^{\infty} e^{-pt}\, sinhat\, dt$

$$\int_0^\infty e^{-pt}\left(\frac{e^{at}-e^{-at}}{2}\right)dt$$

$$\int_0^\infty \left(\frac{e^{-(p-a)t}-e^{-(p+a)t}}{2}\right)dt$$

$$= -\frac{1}{2(p-a)}\left(e^{-\infty}-e^{-0}\right) + \frac{1}{2(p+a)}\left(e^{-\infty}-e^{-0}\right)$$

$$= \frac{1}{2(p-a)} - \frac{1}{2(p+a)}$$

$$= \frac{1}{2}\cdot\frac{2a}{p^2-a^2}$$

Therefore, $L\{sinhat\} = \frac{a}{p^2-a^2}, p > |a|$

Laplace Transformation of derivatives: Let F is an exponential order, and that F is a continuous and f is piecewise continuous on any interval, than $L\{F'(t)\} = \int_0^\infty e^{-pt}F'(t)dt$

$$= [0-F(0)] - \int_0^\infty -pe^{-pt}F(t)dt$$

$$= -F(0) + p\int_0^\infty e^{-pt}F(t)\,dt$$

$$= pL\{F(t)\} - F(0)$$

$$= pf(p) - F(0)$$

Now, since $L\{F'(t)\} = pL\{F(t)\} - F(0)$

Therefore, $L\{F''(t)\} = pL\{F'(t)\} - F'(0)$

$$L\{F''(t)\} = p\{pL\{F(t)\} - F(0)\} - F'(0)$$

$$L\{F''(t)\} = p^2L\{F(t)\} - F(0) - F'(0)$$

$$L\{F''(t)\} = p^2f(p) - F(0) - F'(0)$$

Similarly

$$L\{F'''(t)\} = p^3f(p) - p^2F(0) - pF'(0) - F''(0)$$

and so on……………………..

This is an important part for the solution of differential equations, it is very useful.

Now we will solve differential equation by the Laplace transformation.

Using the method of Laplace transformation to solve the linear differential equation, there are three steps of solving the differential equation by the Laplace transformation of derivative.

(I) Solve the differential equation $\frac{d^2x}{dt^2} = a$ with condition $x(0) = 0, x'(0) = u.$

Solution: We have $x'' = a$

Step-I:

Taking Laplace transformation on both sides

$$L\{x''\} = L\{a\}$$

$$x(p) - px(0) - x'(0) = \frac{a}{p}$$

Step-II:

Simplify to $x(p) = L\{x\}$

$$p^2x(p) - 0 - u = \frac{a}{p}$$

$$x(p) = \frac{a}{p} + u\,x(p) = \frac{a}{p^3} + \frac{u}{p^2}$$

$$L\{x\} = \frac{a}{p^3} + \frac{u}{p^2}$$

Step- III

Find the inverse Laplace transformation

$$x = L^{-1}\{\frac{a}{p^3}\} + L^{-1}\{\frac{u}{p^2}\}$$

Hence the required solution is

$$x = \frac{at^2}{2} + ut$$

(II) Solve $\frac{d^2x}{dt^2} + 9x = cos2t$ with condition $x(0) = 1, x\left(\frac{\pi}{2}\right) = -1.$

Solution: Given equation is

$$+ 9x = cos2t$$

Taking Laplace Transformation on both sides

$$L\{x''\} + 9L\{x\} = L\{cos2t\}$$

$$x(p) - px(0) - x'(0) + 9x(p) = \frac{p}{p^2+4}$$

Using condition $x(0) = 1, x\left(\frac{\pi}{2}\right) = -1$

$$x(p)(p^2+9) - p - A = \frac{p}{p^2+4}$$

$$x(p) = \frac{p+A}{p^2+9} + \frac{p}{(p^2+4)(p^2+9)}$$

$$x(p) = \frac{p+A}{p^2+9} + \frac{p}{(p^2+4)(p^2+9)}$$

By partial fraction

$$x(p) = \frac{p}{p^2+9} + \frac{A}{p^2+9} + \frac{p}{5(p^2+4)} - \frac{p}{5(p^2+9)}$$

$$L\{x\} = \frac{p}{p^2+9} + \frac{A}{p^2+9} + \frac{p}{5(p^2+4)} - \frac{p}{5(p^2+9)}$$

$$x = L^{-1}\left\{\frac{p}{p^2+9}\right\} + L^{-1}\left\{\frac{A}{p^2+9}\right\} + L^{-1}\left\{\frac{p}{5(p^2+4)}\right\} - L^{-1}\left\{\frac{p}{5(p^2+9)}\right\}$$

$$x = cos3t + \frac{A}{3}sin3t + \frac{1}{5}cos2t - \frac{1}{5}cos3t$$

$$x = \frac{A}{3}sin3t + \frac{1}{5}cos2t + \frac{4}{5}cos3t$$

Since $x\left(\frac{\pi}{2}\right) = -1$

$$-1 = -\frac{A}{3} - \frac{1}{5}$$

$$A = \frac{12}{5}$$

Hence the required solution is

$$x = \frac{4}{5}sin3t + \frac{1}{5}cos2t + \frac{4}{5}cos3t$$

(III) Solve

$$+ 2\frac{dx}{dt} - 3x = sint \ with \ condition$$

$$x(0) = x^{'}(0) = 0, at \ t = 0.$$

Solution: The given equation is

$$+ 2x^{'} - 3x = sint$$

Taking Laplace Transformation on both sides

$$L\{x^{''}\} + 2L\{x^{'}\} - 3L\{x\} = L\{sint\}$$

$$p^2x(p) - px(0) - x^{'}(0) + 2px(p) - 2x(0) - 3x(p) = \frac{1}{p^2+1}$$

Using initial conditions

$$(p^2 - 2p - 3)x(p) - 0 - 0 - 0 = \frac{1}{p^2+1}$$

$$x(p) = \frac{1}{(p^2+1)(p^2-2p-3)}$$

$$x(p) = \frac{1}{(p^2+1)(p+3)(p-1)}$$

$$L\{x\} = \frac{1}{(p^2+1)(p+3)(p-1)}$$

$$x = L^{-1}\left\{\frac{1}{(p^2+1)(p+3)(p-1)}\right\}$$

To find the inverse Laplace transformation we assume partial fraction as $\frac{1}{(p^2+1)(p+3)(p-1)} = \frac{A}{(p-1)} + \frac{B}{(p+3)} + \frac{Cp+D}{(p^2+1)}$

$$1 = A(p+3)(p^2+1) + B(p-1)(p^2+1) + (Cp+D)(p-1)(p+3)$$

$$puting \ p = 1 \ then \ A = \frac{1}{8}$$

$$puting \ p = -3 \ then \ B = -\frac{1}{40}$$

$$Equating \ the \ coefficient \ of \ p^3$$

$$A + B + C = 0 \ Therefore \ C = -\frac{1}{10}$$

$$Equating \ the \ constant \ terms$$

$$3A - B - 3D = 1 \ Therefore \ D = -\frac{1}{5}$$

We get,

$$x = L^{-1}\left\{\frac{1}{(p^2+1)(p+3)(p-1)}\right\}$$

$$x = \frac{1}{8}L^{-1}\left\{\frac{1}{(p-1)}\right\} - \frac{1}{40}L^{-1}\left\{\frac{1}{(p+3)}\right\} - \frac{1}{10}L^{-1}\left\{\frac{p}{(p^2+1)}\right\} - \frac{1}{5}L^{-1}\left\{\frac{1}{(p^2+1)}\right\}$$

$$x = \frac{1}{8}e^{t} - \frac{1}{40}e^{-3t} - \frac{1}{10}cost - \frac{1}{5}sint$$

Which is the required solution.

(IV) Solve $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = t \ with \ condition$

$$x(0) = x^{'}(0) = 0, at \ t = 0$$

Solution: The given equation is

$$- 3x^{'} + 2x = t$$

Taking Laplace Transformation on both sides

$$L\{x^{''}\} - 3L\{x^{'}\} + 2L\{x\} = L\{t\}$$

$$p^2x(p) - px(0) - x^{'}(0) - 3px(p) + 3x(0) + 2x(p) = \frac{1}{p^2}$$

Using initial conditions

$$(p^2 - 3p + 2)x(p) - 0 - 0 - 0 = \frac{1}{p^2}$$

$$x(p) = \frac{1}{p^2(p^2-2p-3)}$$

$$x(p) = \frac{1}{4(p-2)} - \frac{1}{p-1} + \frac{3}{4p} + \frac{1}{2p^2}$$

Taking Laplace Transformation on both sides

$$x(t) = L^{-1}\left\{\frac{1}{4(p-2)}\right\} - L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{3}{4p}\right\} + L^{-1}\left\{\frac{1}{2p^2}\right\}$$

$$x(t) = \frac{1}{4}e^{2t} - e^{t} + \frac{3}{4} + \frac{t}{2}$$

Which is the required solution.

Conclusion:

This paper denotes the used of Laplace transformation of elementary function, Laplace transformation of derivatives to find out the particular solution without finding the general solution for the differential equations with boundary conditions. It also used the inverse Laplace transformation and the various examples that can be use in finding the inverse Laplace transformation. It may be finish that this technique is very foremost and accomplished in finding the solution of differential equations.

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2/25/2019