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Abstract: In this paper, the method, Optimal Homotopy Asymptotic Method is applied to solve first order, second order and third order boundary value problems. This method gives a series solution whose convergence is restrained optimally and the convergence region can be adjusted according to the problem concerned. Numerical results are compared with the results obtained by using Modified Adomian decomposition method. The results show that the suggested method is more effective and easy to use.

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1. Introduction

Differential equations can be used to model different physical systems such as sociological, economical, biological and chemical etc. Also in literature physical problems are investigated by differential equations, which are mostly handled by the common methods, variational iteration method (VIM), Adomian decomposition method (ADM), Splines (S), Homotopy perturbation method (HPM). The non perturbed techniques (DTM) and ADM concern nonlinear problems but the region of convergence of their series solution is generally small. Recently Herisanu and Marinca et al. [3-5] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. They used OHAM for

understanding the behavior of nonlinear mechanical vibration of an electrical machine. By using this method they investigated solution of nonlinear equations arising in the study of state flow of a fourth grade fluid past a porous plate. This method supplies the need to control the convergence. In general the OHAM solution agrees with the exact solution. The graphs of the two solutions are coincident. We have applied OHAM to first, second and third order boundary Value problems and have investigated that the original exact solution agrees with the numerical solution, the error noted is small. This method is effective and is easy to use.

2. Analysis of the Method OHAM

Considering the following differential equation

$$L(F(y)) + g(y) + N(F(y)) = 0, B\left(F, \frac{dF}{dy}\right) = 0 \quad (2.1a)$$

Where L is taken as linear operator \mathcal{Y} is independent variable $F(y)$ is an unknown function $g(y)$ is a known function, $N(F(y))$ is a nonlinear operator and B is a boundary operator. According to the idea of OHAM we construct a homotopy as given below

$$H(\theta(y, p), p) : R \times [0, 1] \rightarrow R$$

That satisfies

$$(1-p)[L(\theta(y, p)) + g(y)] = h(p)[L(\theta(y, p)) + g(y) + N(\theta(y, p))],$$

$$B\left(\theta(y, p), \frac{\partial(\theta(y, p))}{\partial y}\right) = 0 \quad (2.2a)$$

Where $y \in R, p \in [0, 1]$ is an embedding parameter, $h(p)$ is a nonzero auxiliary function for $p \neq 0$, $h(0) = 0$ and $\theta(y, p)$ is an Unknown function. Evidently, for $p=0$ and $p=1$ it restrains that Unknown function. Evidently, for $p=0$ and $p=1$ it restrains that $\theta(y, 1) = F(y)$ respectively. Thus as p varies from 0 to 1, the solution $\theta(y, p)$ approaches from $F_0(y)$ to $F_1(y)$ where $F_0(y)$ is obtained from Equation (2.2 a) for $p=0$ and we have.

$$L(F_0(y)) + g(y) = 0, B\left(F_0, \frac{dF_0}{dy}\right) = 0 \quad (2.3a)$$

Now choosing auxiliary function $h(p)$ in the following pattern

$$h(p) = pC_1 + p^2C_2 + \dots \quad (2.4a)$$

Where C_1, C_2, \dots are constants to be determined latter. $h(p)$ can be expressed in many forms as investigated by V. Marinca [3-5]. To get an approximate solution one can expand $\theta(y, p, C_i)$ in Taylor's series about p in the

$$\theta(y, p, C_i) = F_0(y) + \sum_{k=1}^{\infty} F_k(y, C_1, C_2, \dots, C_k) p^k \quad (2.5a)$$

following Pattern.

Making use of equation (2.5a) in equation (2.2a) and comparing the coefficients of like powers of p we have the following linear equations Zeroth order problem is given by equation (2.3a) and the first and second order problems are given by equations (2.6a) and (2.7a)

$$L(F_1(y)) + g(y) = C_1 N_0(F_0(y)), B\left(F_1, \frac{dF_1}{dy}\right) = 0 \quad (2.6a)$$

$$L(F_2(y)) - L(F_1(y)) = C_2 N_0(F_0(y)) + C_1 [L(F_1(y)) + N_1(F_0(y), F_1(y))],$$

$$B\left(F_2, \frac{dF_2}{dy}\right) = 0 \quad (2.7a)$$

The general governing equations for $F_k(y)$ are given by

$$L(F_k(y)) - L(F_{k-1}(y)) = C_k N_0(F_0(y)) + \sum_{i=1}^{k-1} C_i [L(F_{k-i}(y)) + N_{k-i}(F_0(y), F_0(y), \dots, F_{k-i}(y))], k = 2, 3, \dots, B\left(F_k, \frac{dF_k}{dy}\right) = 0 \quad (2.8a)$$

Where $N_m(F_0(y), F_1(y), \dots, F_{k-1}(y))$ is the coefficient of p^m in the expansion of $N(\theta(y, p))$ about the embedding parameter p

$$N(\theta(y, p, C_i)) = N_0(F_0(y)) + \sum_{m=1}^{\infty} N_m(F_0, F_1, \dots, F_m) p^m \quad (2.9a)$$

It has been investigated that the convergence of the series (2.5a) depends on the auxiliary constants C_1, C_2, \dots . If

$$\theta(y, C_i) = F_0(y) + \sum_{k=1}^{\infty} F_k(y, C_1, C_2, \dots, C_k) \quad (2.10a)$$

it is convergent. At $p=1$ then we have

The result of the m th order approximations are

$$\tilde{F}(y, C_1, C_2, \dots, C_m) = F_0(y) + \sum_{i=1}^m F_i(y, C_1, C_2, \dots, C_k) \quad (2.11a)$$

Using equation (2.11a) into equation (2.1a) we get the residual

$$R(y, C_1, C_2, \dots, C_m) = L(\tilde{F}(y, C_1, C_2, \dots, C_m)) + g(y) + N(\tilde{F}(y, C_1, C_2, \dots, C_m)) \quad (2.12a)$$

If $R=0$ then \tilde{F} be the exact solution.

Generally it does not happen especially in nonlinear problems. In order to get the Optimal Values of C_i , $i=1, 2, 3, \dots$ we first construct the Functional Values of C_i , $i=1, 2, 3, \dots$ we first construct the Functional.

$$J(C_1, C_2, \dots, C_m) = \int_a^b R^2(y, C_1, C_2, \dots, C_m) dy \quad (2.13a)$$

And then minimizing it we get

$$\frac{\partial J}{\partial C_1} = 0, \frac{\partial J}{\partial C_2} = 0, \dots, \frac{\partial J}{\partial C_m} = 0 \quad (2.14a)$$

Knowing the values of C_1, C_2, \dots, C_m . The approximate solution of order m is determined. Where a, b lie in domain of the concerned problem using the least square method we get OHAM solution.

3. Numerical Problems

Problem 3.1. [First order linear]

Consider the following linear differential equation

$$F'(y) - F(y) - y \cos y + y \sin y - \sin y = 0, F(0) = 0 \quad (3.1b)$$

The exact solution of the problem is

$$F(y) = y \sin y \quad (3.2b)$$

Applying the method mentioned in section 2, the Zeroth order problem is

$$F_0'(y) = 0, F_0(0) = 0 \quad (3.3b)$$

Its solution is

$$F_0(y) = 0 \quad (3.4b)$$

First order problem is

$$F_1'(y, C_1) = -y \cos y C_1 - \sin y C_1 + y \sin y C_1 - C_1 F_0(y) + (1 + C_1) F_0'(y), \\ F_1(0) = 0 \quad (3.5b)$$

Its solution is

$$F_1(y, C_1) = -y \cos y C_1 + \sin y C_1 - y \sin y C_1 \quad (3.6b)$$

Second order problem is

$$F_2'(y, C_1, C_2) = \left(\begin{array}{l} -y \cos y C_2 - \sin y C_2 + y \sin y C_2 - C_2 F_0(y) - C_1 F_1(y) \\ + C_2 F_0'(y) + (1 + C_1) F_1'(y), F_2(0) = 0 \end{array} \right) \quad (3.7b)$$

$$F_2(y, C_1, C_2) = -y \cos y C_1 + \sin y C_1 - y \sin y C_1 - 2C_1^2 + 2 \cos y C_1^2 \\ - 2y \cos y C_1^2 + 2 \sin y C_1^2 - y \cos y C_2 + \sin y C_2 - y \sin y C_2 \quad (3.8b)$$

Its solution is

Third order problem is

$$F_3'(y, C_1, C_2, C_3) = -y \cos y C_3 - \sin y C_3 + y \sin y C_3 - C_3 F_0(y) - C_2 F_1(y) \\ - C_1 F_2(y) + C_3 F_0'(y) + C_2 F_1'(y) + (1 + C_1) F_2'(y), F_3(0) \quad (3.9b)$$

Its solution is

$$F_3(y, C_1, C_2, C_3) = -y \cos y C_1 + \sin y C_1 - y \sin y C_1 - 4C_1^2 + 4 \cos y C_1^2 - 4y \cos y C_1^2 \\ + 4 \sin y C_1^2 - 6C_1^3 + 2y C_1^3 + 6 \cos y C_1^3 - 2y \cos y C_1^3 + 2y \sin y C_1^3 - y \cos y C_2 \\ + \sin y C_2 - y \sin y C_2 - 4C_1 C_2 + 4 \cos y C_1 C_2 - 4y \cos y C_1 C_2 + 4 \sin y C_1 C_2 \\ - y \cos y C_3 + \sin y C_3 - y \sin y C_3 \quad (3.10b)$$

Now we use equations (3.4b), (3.6b), (3.8b), (3.10b), the third order approximate solution by OHAM for $p=1$ is

$$\tilde{F}(y, C_1, C_2, C_3) = F_0(y) + F_1(y, C_1) + F_2(y, C_1, C_2) + F_3(y, C_1, C_2, C_3) \quad (3.11b)$$

Using the technique mentioned in section 2 on the domain $a=0, b=1$. we use the residual

$$R = \tilde{F}' - \tilde{F} - y \cos y + y \sin y - \sin y \quad (3.12b)$$

The following values of C_1, C_2, C_3 are obtained

$$C_1 = -1.1763684395430054, C_2 = 0.11688800369849017, \\ C_3 = -0.0026970142170896472 \quad (3.13b)$$

Considering the values of C_1, C_2, C_3 , the approximate solution becomes

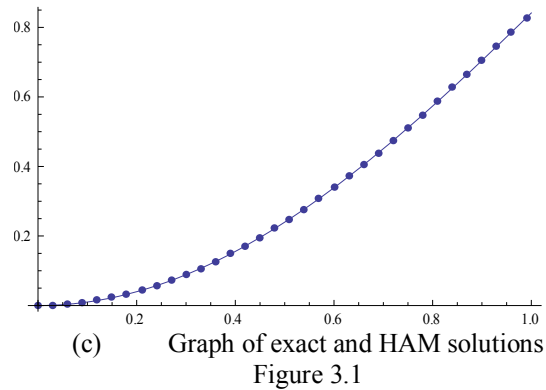
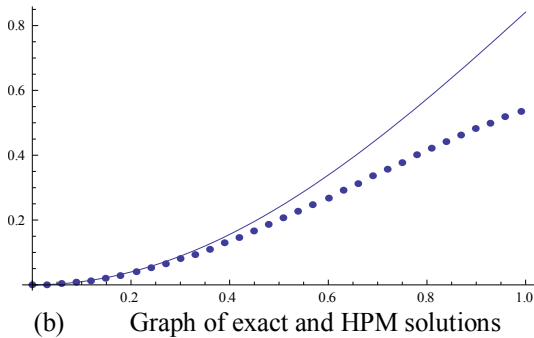
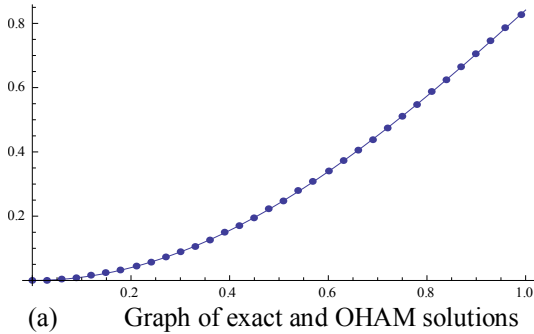
$$\tilde{F}(y) = 1.01231 y^2 - 0.0480322 y^3 - 0.105409 y^4 - 0.0223286 y^5 + 0.00421533 y^6 \\ + 0.00112045 y^7 - 0.0000878039 y^8 - 0.0000237398 y^9 + 1.11482 \times 10^{-6} y^{10} \\ + 2.90161 \times 10^{-7} y^{11} - 9.50035 \times 10^{-9} y^{12} - 2.33658 \times 10^{-9} y^{13} + 5.79949 \times 10^{-11} y^{14} \\ + 1.3396 \times 10^{-11} y^{15} + O(y^{16}) \quad (3.14a)$$

Table 3.1 displays values of exact solution (3.2b) OHAM solution (3.14b) and the error.

y	Exact sol	OHAM sol	HPM sol	HAM sol	Er OHAM	Er HPM	Er HAM
0.0	0.000000	0.000000	0.000000	0.0000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	0.00998334	0.0100643	0.00965034	0.00999199	8.0 E-6	3.3 E-4	8.0 E-6
0.2	0.0397379	0.0399325	0.0370779	0.0397289	1.9 E-4	2.6 E-3	4.9 E-6
0.3	0.0886561	0.0889059	0.0797368	0.0886099	2.4 E-4	8.9 E-3	4.6 E-5
0.4	0.155767	0.155987	0.134773	0.155695	2.1 E-4	2.0 E-2	7.2 E-5
0.5	0.239713	0.239861	0.199079	0.239691	1.4 E-4	4.0 E-2	2.1 E-5

y	Exact sol	OHAM sol	HPM sol	HAM sol	Er OHAM	Er HPM	Er HAM
0.6	0.33785	0.333884	0.269344	0.338939	9.8 E-5	6.9 E-2	1.5 E-4
0.7	0.450952	0.451076	0.342124	0.451411	1.2 E-4	1.0 E-1	4.5 E-4
0.8	0.573885	0.574113	0.413894	0.574712	2.2 E-4	1.5 E-1	8.2 E-4
0.9	0.704994	0.705338	0.481116	0.706085	3.4 E-4	2.2 E-1	1.0 E-3
1.0	0.841471	0.841762	0.540302	0.842428	2.9 E-4	3.0 E-1	9.5 E-4

In the Table 3.1 the values of HPM, HAM and their errors are also given. From the table given above we conclude that the errors of the technique, OHAM are smaller than the errors of HPM and HAM.



From the figure 3.1(a) we investigate that the graphs of OHAM and exact solutions coincide while in figure 3.1(b) the graphs of HPM and exact solutions are not coincident and the graphs of figure 3.1(c) are again coinciding but generally the graphs of OHAM are in best agreement with their exact solutions. From the table 3.1 given above we reach to the conclusion that the exact and OHAM solutions are in best agreement in the domain $a=0, b=1$, Solid Curve= exact solution, Dotted curve= OHAM solution. We conclude that the two Curves coincide.

Problem 3.2: [Second order linear]

For $y \in [0,1]$ we consider the following differential equation $F''(y) + 2F'(y) + F(y) = 0, F(0) = 1, F'(0) = 0$ (3.1c)

The exact solution of the problem is

$$F(y) = e^{-y} + ye^{-y} \tag{3.2c}$$

Applying the technique, OHAM our Zeroth order problem is

$$F_0''(y) + 2F_0'(y) + F_0(y) = 0, F_0(0) = 1, F_0'(0) = 0 \tag{3.3c}$$

Its solution is $F_0(y) = \cos y$ (3.4c)

First order problem is

$$F_1''(y, C_1) = -F_1(y) - 2F_1'(y) + (1 + C_1)[F_0(y) + 2F_0'(y) + F_0''(y)], F_1(0) = 0, F_1'(0) = 0 \tag{3.5c}$$

Its solution is

$$F_1(y, C_1) = \frac{1}{2} \begin{pmatrix} 2y \cos y - \sin y + \cos 2y \sin y - \cos y \sin 2y + 2y \cos y C_1 \\ -\sin y C_1 + \cos 2y \sin y C_1 - \cos y \sin 2y C_1 \end{pmatrix} \tag{3.6c}$$

Second order problem is

$$F_2''(y, C_1, C_2) = \begin{pmatrix} C_2 F_0'(y) - F_2(y) + 2C_2 F_0'(y) - 2F_2'(y) + C_2 F_0''(y) \\ + (1 + C_1)[F_1(y) + 2F_1'(y) + F_1''(y)] \\ F_2(0) = 0, F_2'(0) = 0 \end{pmatrix} \quad (3.7c)$$

Its solution is

$$F_2(y, C_1, C_2) = \frac{1}{4} \begin{pmatrix} \cos y + 4y \cos y + 2y^2 \cos y - \cos y \cos 2y - 2 \sin y \\ + 2 \cos 2y \sin y + 2y \cos 2y \sin y - 2 \cos y \sin 2y \\ - 2y \cos y \sin 2y - \sin y \sin 2y + 2 \cos y C_1 \\ + 8y \cos y C_1 + 4y^2 \cos y C_1 - 2 \cos y \cos 2y C_1 \\ - 4 \sin y C_1 + 4 \cos 2y \sin y C_1 + 4y \cos 2y \sin y C_1 \\ - 4 \cos y \sin 2y C_1 - 4y \cos y \sin 2y C_1 \\ - 2 \sin y \sin 2y C_1 + \cos y C_1^2 + 4y \cos y C_1^2 \\ + 2y^2 \cos y C_1^2 - \cos y \cos 2y C_1^2 - 2 \sin y C_1^2 \\ + 2 \cos 2y \sin y C_1^2 + 2y \cos 2y \sin y C_1^2 \\ - 2 \cos y \sin 2y C_1^2 - 2y \cos y \sin 2y C_1^2 \\ - \sin y \sin 2y C_1^2 + 4y \cos y C_2 - 2 \sin y C_2 \\ + 2 \cos 2y \sin y C_2 - 2 \cos y \sin 2y C_2 \end{pmatrix} \quad (3.8c)$$

Third order problem is

$$F_0'''(y, C_1, C_2, C_3) = \begin{pmatrix} C_3 F_0'(y) + C_2 F_1'(y) - F_3(y) + 2C_3 F_0''(y) \\ + 2C_2 F_1''(y) - 2F_3'(y) + C_3 F_0'''(y) \\ + C_2 F_1'''(y)(1 + C_1)[F_2(y) + 2F_2'(y) + F_2''(y)], \\ F_3(0) = 0, F_3'(0) = 0 \end{pmatrix} \quad (3.9c)$$

Its solution is

$$F_3(y, C_1, C_2, C_3) = \frac{1}{2} \begin{pmatrix} C_2 F_0'(y)[- \cos y - 2y \cos y + 2 \cos^2 y - \cos y \cos 2y \\ + \sin y - 2y \sin y + 2 \cos^2 y \sin y - \cos 2y \sin y \\ + 2 \sin^2 y - \sin y \sin 2y] + C_1 C_2 F_0'(y)[- \cos y - 2y \cos y \\ + 2 \cos^2 y - \cos y \cos 2y + \sin y - 2y \sin y + 2 \cos^2 y \sin y \\ - \cos 2y \sin y + 2 \sin^2 y - \sin y \sin 2y] + C_3 F_0'(y)[- 2 \cos y \\ + 2 \cos^2 y + 2 \sin^2 y] + F_1(y)[- \cos y - 2y \cos y + 2 \cos^2 y \\ - \cos y \cos 2y + \sin y - 2y \sin y + 2 \cos^2 y \sin y \\ - \cos 2y \sin y + 2 \sin^2 y - \sin y \sin 2y] + C_1 F_1'(y)[- 2 \cos y \\ - 4y \cos y + 4 \cos^2 y + 2 \cos y \cos 2y + 2 \sin y - 4y \sin y \\ + 4 \cos^2 y \sin y - 2 \cos 2y \sin y + 4 \sin^2 y - 2 \sin y \sin 2y] \\ + C_1^2 F_1'(y)[- \cos y - 2y \cos y + 2 \cos^2 y - \cos y \cos 2y \\ + \sin y - 2y \sin y + 2 \cos^2 y \sin y - \cos 2y \sin y + 2 \sin^2 y \\ - \sin y \sin 2y] + C_2 F_1'(y)[- 2 \cos y + 2 \cos^2 y + 2 \sin^2 y] \end{pmatrix} \quad (3.10c)$$

(3.8c) and (3.10c) we get third order approximate OHAM solution for $P=1$

$$\tilde{F}(y, C_1, C_2, C_3) = F_0(y) + F_1(y, C_1) + F_2(y, C_1, C_2) + F_3(y, C_1, C_2, C_3) \quad (3.11c)$$

We use equations (3.4c), (3.6c),

Using the OHAM technique of section mentioned above on domain $a=0, b=1$ we use the residual R that is $R = \tilde{F}''(y) + 2\tilde{F}'(y) + \tilde{F}(y)$ (3.12c)

We have obtained the following values of C_1, C_2, C_3 . where $C_1 = -0.6157891011374226, C_2 = -1.648024704278988,$
 $C_3 = -0.48317914168990783$ (3.13c)

From the above values of C_i 's we get the following approximate solution $\tilde{F}(y) = 1 - 0.424995y^2 - 0.161002y^3$ (3.14c)

The following table 3.4 displays values of the exact solution (3.2c), OHAM solution (3.14c) and Error of OHAM. We compare the two solutions there exists similarity approximations between them also the values of HPM, HAM along with their errors are considered and the comparison is established between the errors of OHAM, HPM and HAM the errors of the technique, OHAM are smaller than the other two. All the mentioned values are described in the following table 3.2, their graphs are drawn in figure 3.2 below.

Figure 3.2

y	Exact sol	OHAM sol	HPM sol	HAM sol	Er OHAM	Er HPM	Er HAM
0.0	1.000000	1.000000	1.000000	1.000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	0.995321	0.99591	0.995338	0.995562	5.9 E-4	1.6 E-5	2.4 E-4
0.2	0.982477	0.984288	0.982733	0.983077	1.8 E-3	2.5 E-5	5.9 E-4
0.3	0.963064	0.966098	0.964338	0.963763	3.0 E-3	1.2 E-3	6.9 E-4
0.4	0.938448	0.942305	0.9424	0.938806	3.8 E-3	3.9 E-3	3.5 E-4
0.5	0.909796	0.923877	0.919271	0.90935	4.0 E-3	9.4 E-3	4.4 E-4
0.6	0.878099	0.881778	0.8974	0.876489	3.6 E-3	1.9 E-2	1.6 E-3
0.7	0.844195	0.846976	0.879337	0.841258	2.7 E-3	3.5 E-2	2.9 E-3
0.8	0.808792	0.810436	0.867733	0.804626	1.6 E-3	5.8 E-2	4.1 E-3
0.9	0.772482	0.773125	0.865338	0.76749	6.4 E-4	9.2 E-2	4.9 E-3
1.0	0.735759	0.736007	0.875	0.73066	2.4 E-4	1.3 E-1	5.0 E-3

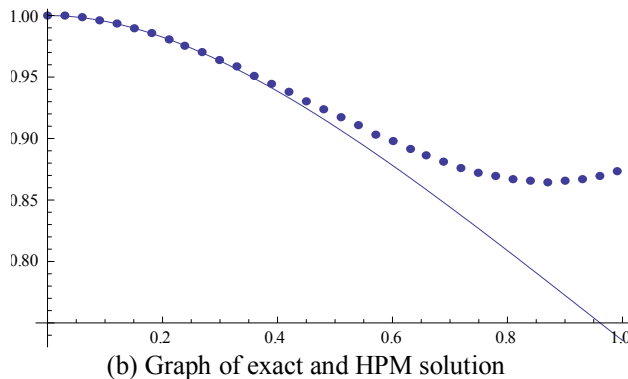
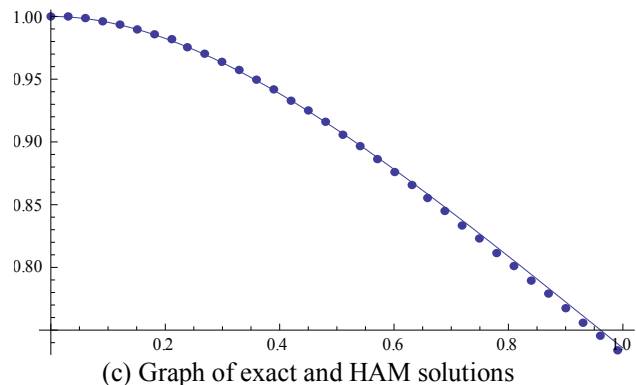
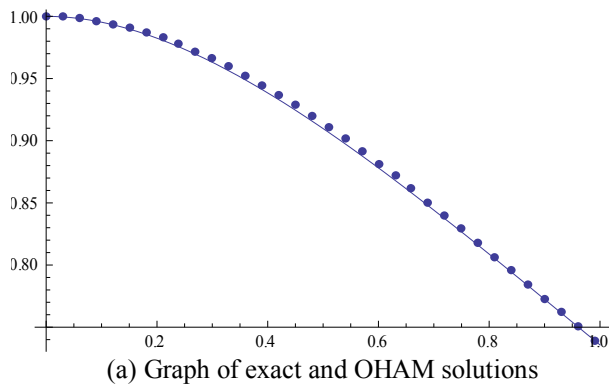


Figure 3.2

From the Figure 3.2(a) We conclude that the two graphs are coincident and also from the above Table 3.2 We investigate that the OHAM solution and exact solution are in very good agreement on domain $a=0, b=1$ showing that the method is effective. Solid curve = exact solution Dotted curve= OHAM solution, HPM solution and HAM solution. From figure 3.2(b) the HPM and exact solutions are not coincident again the HAM and exact solutions are coincident as shown in figure 3.2(c), However generally the OHAM errors

are smaller and there exists accuracy in the values of OHAM and the method is reliable and effective.

For $y \in [0,1]$ we consider the following linear differential equation

Problem 3.3: [third order linear]

$$F'''(y) + F(y) - (7 - y^2)\cos y - (y^2 - 6y - 1)\sin y = 0,$$

$$F(0) = 0, F'(0) = -1, F'(1) = 2\sin(1) \quad (3.1d)$$

The exact solution of the problem is $F(y) = (y^2 - 1)\sin y$ (3.2d)

Applying the technique of OHAM that is described in the above section. The Zeroth order problem is

$$F_0'''(y) = 0, F_0(0) = 0, F_0'(0) = -1, F_0'(1) = 2\sin(1) \quad (3.3d)$$

Its solution is

$$F_0(y) = \frac{1}{2}(-2y + y^2(1 + 2\sin(1))) \quad (3.4d)$$

First order problem is

$$F_1'''(y, C_1) = \left(\begin{array}{l} C_1[-7\cos y + y^2\cos y + \sin y + 6y\sin y - y^2\sin y + F_0(y)] \\ +(1 + C_1)F_0'''(y), F_1(0) = 0, F_1'(0) = 0, F_1'(1) = 0 \end{array} \right) \quad (3.5d)$$

Its solution is

$$F_1(y, C_1) = \frac{1}{240}C_1 \left(\begin{array}{l} -3120 - 240y + 135y^2 - 10y^4 + 2y^5 - 480y^2\cos(1) \\ +3120\cos y - 240y^2\cos y + 950y^2\sin(1) + 4y^5\sin(1) \\ +240\sin y + 1440y\sin y - 240y^2\sin y \end{array} \right) \quad (3.6d)$$

Second order problem is

$$F_2'''(y, C_1, C_2) = \left(\begin{array}{l} C_2[-7\cos y + y^2\cos y + \sin y + 6y\sin y - y^2\sin y] \\ +F_0(y) + F_0'''(y) + C_1F_1(y) + (1 + C_1)F_1'''(y), \\ F_2(0) = 0, F_2'(0) = 0, F_2'(1) = 0 \end{array} \right) \quad (3.7d)$$

Its solution is

$$F_2(y, C_1, C_2) = \frac{1}{40320} \left(\begin{array}{l} C_1[-52416 - 40320y + 22680y^2 - 1680y^4 + 336y^5 \\ -80640y^2\cos(1) + 524160\cos y - 40320y^2\cos y \\ +159600y^2\sin(1) + 672y^5\sin(1) + 40320\sin y \\ +241920y\sin y - 40320y^2\sin y] + C_1^2[-1048320 \\ +1209600y - 468801y^2 - 87360y^3 - 3360y^4 \\ +714y^5 - 8y^7 + y^8 + 446880y^2\cos(1) - 1344y^5\cos(1) \\ +1048320\cos y + 483840y\cos y - 80640y^2\cos y \\ +475502y^2\sin(1) + 3332y^5\sin(1) + 2y^8\sin(1) \\ -1693440\sin y + 483840y\sin y] + C_2[-524160 \\ -40320y + 22680y^2 - 1680y^4 + 336y^5 - 80640y^2\cos(1) \\ +524160\cos y - 40320y^2\cos y + 159600y^2\sin(1) \\ +672y^5\sin(1) + 40320\sin y + 241920y\sin y \\ -40320y^2\sin y] \end{array} \right) \quad (3.8d)$$

$$F_3'''(y, C_1, C_2, C_3) = \left(\begin{array}{l} C_3[-7\cos y + y^2\cos y + \sin y + 6y\sin y - y^2\sin y] \\ +F_0(y) + F_0'''(y) + C_2[F_1(y) + F_1'''(y)] + C_1F_2(y) \\ +(1 + C_1)F_2'''(y), F_3(0) = 0, F_3'(0) = 0, F_3'(1) = 0 \end{array} \right) \quad (3.9d)$$

Third order problem is

Its solution is obtained by using the method, OHAM. The optimal values of the auxiliary constants C_1, C_2, C_3 are obtained by using Galerkin's method or least square method, using these values we get the series solution.

$$F_3(y, C_1, C_2, C_3) = \frac{1}{159667200} \left(\begin{aligned} & C_1[-2075673600 - 159667200y \\ & + 89812800y^2 - 6652800y^4 \\ & + 1330560y^5 - 319334400y^2 \\ & \cos(1) + 2075673600 \cos y \\ & - 159667200y^2 \cos y + 632016000y^2 \\ & \sin(1) + 2661120y^5 \sin(1) + 159667200 \\ & \sin y + 958003200y \sin y - 159667200y^2 \\ & \sin y] + C_1^2[-8302694400 + 9580032000y \\ & - 3712903920y^2 - 691891200y^3 \\ & - 26611200y^4 + 5654880y^5 - 63360y^7 \\ & + 7920y^8 + 3539289600y^2 \cos(1) \\ & - 10644480y^5 \cos(1) + 8302694400 \cos y \\ & + 3832012800y \cos y - 638668800y^2 \cos y \\ & + 3765975840y^2 \sin(1) + 26389440y^5 \\ & \sin(1) + 15840y^8 \sin(1) - 13412044800 \\ & \sin y + 3832012800y \sin y] + C_1^3[8302694400 \\ & + 14689382400y - 6081277257y^2 \\ & - 1037836800y^3 + 186278400y^4 \\ & - 28113426y^5 - 2882880y^6 - 95040y^7 \\ & + 12375y^8 - 44y^{10} + 4y^{11} + 7443992160y^2 \\ & \cos(1) + 24171840y^5 \cos(1) - 15840y^8 \cos(1) \\ & - 8302694400 \cos y + 5748019200y \cos y \\ & - 319334400y^2 \cos y - 1867972634y^2 \\ & \sin(1) + 44577852y^5 \sin(1) + 47190y^8 \\ & \sin(1) + 8y^{11} \sin(1) - 20437401600 \sin y \\ & + 319334400y^2 \sin y] + C_2[-2075673600 \\ & + 89812800y^2 - 6652800y^4 + 1330560y^5 \\ & - 319334400y^2 \cos(1) + 2075673600 \cos y \\ & - 159667200y^2 \cos y + 632016000y^2 \sin(1) \\ & + 2661120y^5 \sin(1) + 159667200 \sin y \\ & + 958003200y \sin y - 159667200y^2 \sin y] \\ & + C_1 C_2[-8302694400 + 9580032000y \\ & - 3712903920y^2 - 691891200y^3 \\ & - 26611200y^4 + 565880y^5 - 63360y^7 \\ & + 7920y^8 + 3539289600y^2 \cos(1) \\ & - 10644480y^5 \cos(1) + 8302694400 \cos y \\ & + 3832012800y \cos y - 638668800y^2 \\ & \cos y + 3765975840y^2 \sin(1) + 26389440y^5 \\ & \sin(1) + 15840y^8 \sin(1) - 13412044800 \sin y \\ & + 3822012800y \sin y] + C_3[-2075673600 \\ & - 159667200y + 89812800y^2 - 6652800y^4 \\ & + 1330560y^5 - 319334400y^2 \cos(1) \\ & + 2075673600 \cos y + 159667200y^2 \cos y \\ & + 632016000y^2 \sin(1) + 2661120y^5 \sin(1) \\ & + 159667200 \sin y + 958003200y \sin y \\ & - 159667200y^2 \sin y] \end{aligned} \right) \quad (3.10d)$$

Now we use equations (3.4d), (3.6d), (3.8d) and (3.10d) to get third order approximate solution by OHAM for $p=1$ that is

$$\tilde{F}(y, C_1, C_2, C_3) = F_0(y) + F_1(y, C_1) + F_2(y, C_1, C_2) + F_3(y, C_1, C_2, C_3) \tag{3.11d}$$

Using the proposed technique of section described above on the domain $a=0, b=1$ we use the residual $R = \tilde{F}'''(y) + \tilde{F}(y) - (7 - y^2) \cos y - (y^2 - 6y - 1) \sin y$ (3.12d)

The following values of C_i 's are found

$$C_1 = 1.062670102836802, C_2 = -2.208103637790291, C_3 = 0.33369993779942651$$

We use the above values of C_1, C_2, C_3 , the approximate solution is

$$\begin{aligned} \tilde{F}(y) = & -y + 0.0000488077y^2 + 1.16623y^3 - 0.173884y^5 \\ & - 0.00092637y^6 + 0.00852856y^7 + 0.000200624y^8 \\ & - 0.000245363y^9 - 1.12908 \times 10^{-6}y^{10} + 4.06776 \times 10^{-6}y^{11} \\ & - 3.01577 \times 10^{-9}y^{12} - 4.10508 \times 10^{-8}y^{13} + 1.67752 \times 10^{-10}y^{14} \\ & + 2.89409 \times 10^{-10}y^{15} + O(y^{16}) \end{aligned} \tag{3.13d}$$

The following table 6.5 displays values of OHAM solution, exact solution and error

Table 3.3

y	Exact sol	OHAM sol	HPM sol	HAM sol	Er OHAM	Er HPM	Er HAM
0.0	0.000000	0.000000	0.000000	0.000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	-0.0988351	-0.0886448	-0.0988249	-0.0988314	1.9 E-4	1.0 E-5	3.6 E-6
0.2	-0.190723	-0.190034	-0.190682	-0.19071	6.8 E-4	4.0 E-5	1.2 E-5
0.3	-0.268923	-0.267535	-0.268833	-0.268899	1.3 E-3	9.0 E-5	2.4 E-5
0.4	-0.327111	-0.32492	-0.326954	-0.327074	2.1 E-3	1.5 E-4	3.7 E-5
0.5	-0.359569	-0.357602	-0.35933	-0.35952	3.0 E-3	2.3 E-4	4.9 E-5
0.6	-0.361371	-0.357602	-0.361042	-0.36131	3.7 E-3	3.2 E-4	6.0 E-5
0.7	-0.328551	-0.324145	-0.328131	-0.32848	4.4 E-3	4.2 E-4	7.1 E-5
0.8	-0.258248	-0.253368	-0.257746	-0.258169	4.8 E-3	5.0 E-4	7.9 E-5
0.9	-0.148832	-0.143667	-0.148271	-0.148747	5.1 E-3	5.6 E-4	8.4 E-5
1.0	0.000000	0.0052575	0.00058439	0.0000869	5.2 E-3	5.8 E-4	8.6 E-5

Also in the above Table 3.3 the values of HPM and HAM solutions along with their errors are displayed and we conclude that the errors of technique, OHAM are smaller than HPM and HAM solutions. From the above table 3.3 we conclude that OHAM and exact solutions are in best agreement and the values of the two columns are nearly equal.

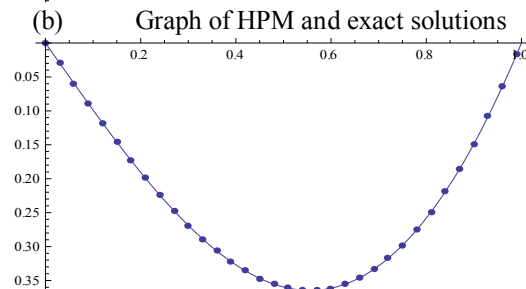
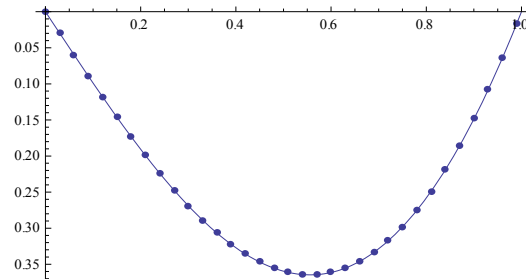
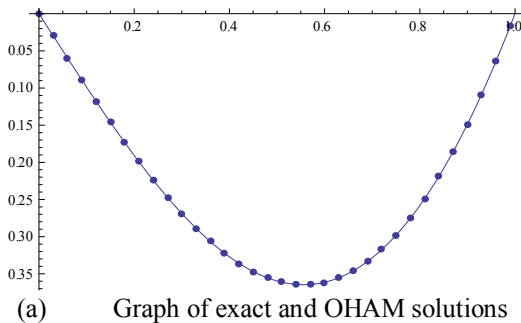


Figure 3.3

In figure 3.3(a), (b) and (c) we investigate that the graphs of OHAM, HPM and HAM are coincident with the Graphs of their exact solutions, but generally we conclude that the technique, OHAM is more effective than the other two. From the above figure 3.3 we investigate that two graphs that is exact and OHAM solution graphs are coincident, this shows that the method OHAM is effective and reliable. Solid curve= exact solution, Dotted curve= OHAM solution, HPM solution and HAM solution.

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