Recurrence Relations for Single Moments of Order Statistics from Independent Non-identically Distributed Generalized Inverted Exponential Variables

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Abstract: The order statistics (OS) arising from independent non-identically (INID) distributed two parameter Generalized Inverted Exponential (GIE) random variables is discussed in this article. Special cases of the rth OS are considered. The moments of the rth order statistic arising from INID distributed GIE random variables are computed. Graphical representations of the probability density function (pdf) and the cumulative distribution function (cdf) of the rth OS arising from INID GIE distribution are plotted. Finally, some numerical examples are given.

Key words: Moments, Non-identically Order Statistics, Generalized Inverted Exponential Distribution.

1-　　Introduction

Generalized Inverted Exponential distribution (GIED) was first introduced by **[3],** which is a generalized form of the inverted exponential distribution (IED) See: **[12]** & **[15].** The important of GIED in statistical and reliability was discussed in **[3], [5]** and **[6].**

The probability density function pdf of two-parameter GIED is given by

$$f(x) = \left(\frac{\theta \alpha}{x^2} \right) e^{-\frac{\alpha}{x}} \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta - 1}, \ x > 0, \ \alpha, \theta > 0 \qquad (1)$$

The cumulative distribution function cdf of two - parameter GIED is given by

$$F(x) = 1 - \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta}, \ x > 0, \ \alpha, \theta > 0 \qquad (2)$$

The maximum likelihood estimation and least square estimation are used in **[3]** to evaluate the parameters and the reliability of the distribution.

[17] obtained the maximum likelihood and Bayes estimators of the parameters of the GIED in case of the progressive type-II censoring scheme with binomial removals. **[13]** generalized the two parameter GIED using the quadratic rank transmutation map. He discussed the properties of the GIED, derived the moments and examined the order statistics. Moreover, the maximum likelihood estimators for the parameters is briefly investigated and the information matrix is derived. **[16]** considered various methods of estimation of the unknown parameters of a generalized inverted exponential distribution from a frequentist as well as Bayesian perspective.

[5] handled the reliability analysis of extended GIED with applications. **[6]** Considered the four-parameter beta GIED and investigated various properties of the model with graphs.

The motive behind this research is that the study of the order statistics for independent but non-identically variables follows generalized inverted exponential distribution was not previously examined, as well as the recurrence relationships to the single moments of the rth order statistics from this distribution was not previously introduced, in which it is difficult to find a closed form.

In section 2, we obtained the probability density function pdf and the cumulative distribution function cdf of the rth OS arising from INID GIED and their graphical representation at n = 3. Computation of moments of the rth OS of INID random variables arising from GIED are found in section 3,

2-   Non-identical order statistics from GIED:

The subject on non-identical order statistics is explained widely in the literature in [1], [2], [7], [8], [9], [10] and [18] for continuous distributions. Also, in [11] for discrete distributions.

From [8], [10] and [18], the pdf & cdf of the rth INID OS written as:

$$f_{r:n}(x) = \frac{1}{(r-1)!\,(n-r)!} \sum_{p} \prod_{a=1}^{r-1} F_{i_a}(x)\, f_{i_r}(x) \prod_{c=r+1}^{n} \left\{ 1 - F_{i_c}(x) \right\} \qquad (3)$$

Where $\sum_{p}$ denotes the summation over all $n!$ permutations $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots n)$. [8] write it in the form of Permanent as:

$$f_{r:n}(x) = \frac{1}{(r-1)!\,(n-r)!} \; per \left[\underbrace{F(x)}_{r-1} \quad \underbrace{f(x)}_{1} \quad \underbrace{\{1-F(x)\}}_{n-r} \right] \qquad (4)$$

$$F_{(r)}(x) = \sum_{j=r}^{n} \sum_{p_j} \prod_{a=1}^{j} F_{i_a}(x) \prod_{a=j+1}^{n} \left[1 - F_{i_a}(x) \right] \qquad (5)$$

Where $\sum_{p_j}$ is all permutations of $(i_1, i_2, \cdots, i_n)$ for $(1, \cdots, n)$ which satisfy $i_1 < i_2 < \cdots < i_j$ and $i_{j+1} < i_{j+2} < \cdots < i_n$. Using Permanent, equation (5) written as

$$F_{(r)}(x) = \sum_{i=r}^{n} \frac{1}{i!\,(n-i)!} \; per \begin{bmatrix} F_1(x) & 1-F_1(x) \\ \vdots & \\ \underbrace{F_n(x)}_{i} & \underbrace{1-F_n(x)}_{n-i} \end{bmatrix}, \quad -\infty < x < \infty \qquad (6)$$

For GIE distribution where:

$$F_i(x) = 1 - \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta_i}, \; x > 0, \; \alpha > 0, \theta_i > 0 \qquad (7)$$

$$f_i(x) = \theta_i\, \alpha^2\, x^{-2} \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta_i - 1} e^{-\frac{\alpha}{x}}, \; x > 0, \; \alpha > 0, \theta_i > 0 \qquad (8)$$

If we substituting Eq. (7) and (8) in Eq. (4) and (6) and using Mathematica Program 11, we obtain special cases from pdf and cdf of the first, second & the last order statistic when n = 3 as:

$$f_{1:3}(x) = \frac{\alpha \sum_{i=1}^{3} \theta_i}{x^2} \; e^{-\frac{\alpha}{x}} (1 + e^{-\frac{\alpha}{x}})^{\sum_{i=1}^{3} \theta_i - 1}$$

$$f_{2:3}(x) = \frac{\alpha}{x^2} e^{-\frac{\alpha}{x}} \left(\sum_{1 \le i_1 < i_2 \le 3} (1 - e^{-\frac{\alpha}{x}})^{(\theta_{i_1} + \theta_{i_2}) - 1} (\theta_{i_1} + \theta_{i_2}) - 2 \sum_{i=1}^{3} \theta_i (1 - e^{-\frac{\alpha}{x}})^{\sum_{i=1}^{3} \theta_i - 1} \right)$$

$$f_{3:3}(x) = \frac{\alpha}{x^2} e^{-\frac{\alpha}{x}} \left(\begin{array}{c} \sum_{i=1}^{3} \theta_i (1-e^{-\frac{\alpha}{x}})^{\theta_i - 1} + \sum_{i=1}^{3} \theta_i (1-e^{-\frac{\alpha}{x}})^{\sum_{i=1}^{3} \theta_i - 1} \\ + \sum_{1 \leq i_1 < i_2 \leq 3} (\theta_{i_1} + \theta_{i_2})(1-e^{-\frac{\alpha}{x}})^{\sum_{1 \leq i_1 < i_2 \leq 3} (\theta_{i_1} + \theta_{i_2}) - 1} \end{array} \right)$$
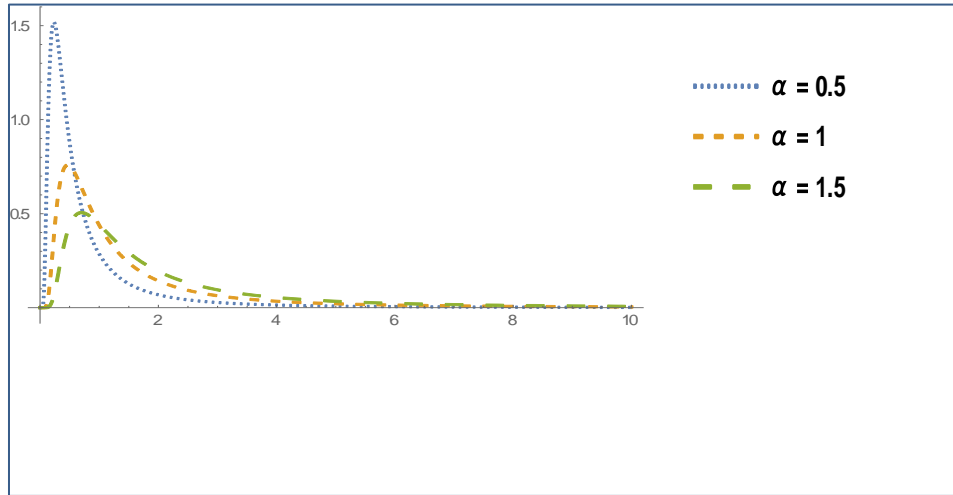

Figure1: Graph of pdf of the first INID OS at $\theta_1 = 0.3$, $\theta_2 = 0.5$, $\theta_3 = 0.7$ & $\alpha = \{ 0.5, 1, 1.5 \}$

From Figure (1), we see that the pdf of the first INID GIED is unimodal for different values of the parameters α.

The special cases of cdf when n = 3:

$$F_{1:3}(x) = 1 - (1-e^{-\frac{\alpha}{x}})^{\theta_1 + \theta_2 + \theta_3}$$

$$F_{2:3}(x) = 1 - (1-e^{-\frac{\alpha}{x}})^{\theta_1 + \theta_2} - (1-e^{-\frac{\alpha}{x}})^{\theta_1 + \theta_2}$$
$$- (1-e^{-\frac{\alpha}{x}})^{\theta_1 + \theta_2} + 2 (1-e^{-\frac{\alpha}{x}})^{\theta_1 + \theta_1 + \theta_2}$$

$$F_{3:3}(x) = (1 - (1-e^{-\frac{\alpha}{x}})^{\theta_1}) (1 - (1-e^{-\frac{\alpha}{x}})^{\theta_2}) (1 - (1-e^{-\frac{\alpha}{x}})^{\theta_3})$$
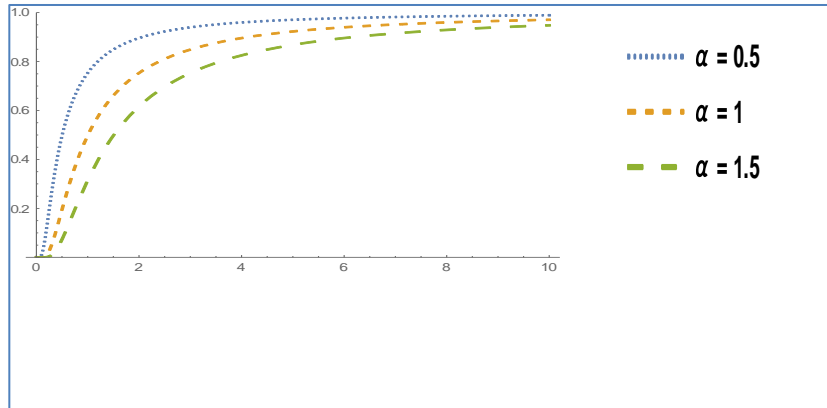
Figure 2: Graph of cdf of the first INID OS at $\theta_1 = 0.3$, $\theta_2 = 0.5$, $\theta_3 = 0.7$ & $\alpha =\{ 0.5,\ 1,\ 1.5 \}$

3- The Recurrence Relations of the Single Moments of the rth OS arising from INID GIE Random Variables

In this section, we shall present recurrence relation for the single moments of INID OS arising from Eq. (7) and the theorem of (Barakat & Abdelkader, 2003). See [**9**].

Theorem 3.1 :

For $1 \le r \le n$, $k = 1,\ 2,\ \cdots$

$$I_j(k) = k\ \alpha^k\ \Big(\sum_{\substack{i=0 \\ i \ne k}}^{\infty} \frac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \Big) \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \sum_{a_1=0}^{\theta_{i_1}} \cdots \sum_{a_j=0}^{\theta_{i_j}} \psi_{i_j} \Big(\sum_{t=1}^{j} a_t \Big)^k \qquad (9)$$

Where:

$$\psi_{i_j} = \prod_{t=1}^{j} \frac{\Gamma(\theta_{i_t}+1)}{\Gamma(\theta_{i_t}-a_t+1)} \frac{(-1)^{a_t}}{a_t\,!}$$

Or

$$\psi_{i_j} = \prod_{t=1}^{j} \frac{(-1)^{a_t}}{(\theta_{i_t}+1)\,\beta\,(a_t+1,\theta_{i_t}-a_t+1)}$$

Proof:

In [**9**] Barakat & Abdelkader derived the single moments of INID OS and generalize this recurrence relation:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \qquad (10)$$

Where:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} G_{i_t}(x)\,dx,\ \ j=1,2,\cdots,n \qquad (11)$$

Where $G_{i_t}(x) = 1 - F_{i_t}(x)$, with $\big(i_1,i_2,\cdots,i_n\big)$ is a permutation of (1, 2, …, n) for which $i_1 \le i_2 < \cdots < i_n$.

Now

Let X_1 , X_2 , $\cdots$, X_n are INID OS from GIED, substituting Eq. (7) in Eq. (11), we obtain:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^\infty x^{k-1} \prod_{t=1}^j \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta_{i_t}} dx, \qquad (12)$$

Using this series: See **[14]**

$$(1+z)^b = \sum_{a=0}^b \frac{\Gamma(b+1)}{\Gamma(b-a+1)} \frac{z^a}{a!}$$

$$\therefore \left[1 - e^{-\frac{\alpha}{x}} \right]^{\theta_{i_t}} = \sum_{a=0}^{\theta_{i_t}} \frac{\Gamma(\theta_{i_t}+1)}{\Gamma(\theta_{i_t}-a+1)} \frac{(-1)^a e^{-a\frac{\alpha}{x}}}{a!}$$

$$\therefore I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^\infty x^{k-1} \prod_{t=1}^j \sum_{a=0}^{\theta_{i_t}} \frac{\Gamma(\theta_{i_t}+1)}{\Gamma(\theta_{i_t}-a+1)} \frac{(-1)^a e^{-a\frac{\alpha}{x}}}{a!} dx,$$

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^\infty x^{k-1} \left(\sum_{a_1=0}^{\theta_{i_1}} \frac{\Gamma(\theta_{i_1}+1)}{\Gamma(\theta_{i_1}-a_1+1)} \frac{(-1)^{a_1} e^{-a_1\frac{\alpha}{x}}}{a_1!} \times \cdots \right.$$

$$\left. \times \sum_{a_j=0}^{\theta_{i_j}} \frac{\Gamma(\theta_{i_j}+1)}{\Gamma(\theta_{i_j}-a_j+1)} \frac{(-1)^{a_j} e^{-a_j\frac{\alpha}{x}}}{a_j!} dx \right),$$

$$\therefore I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^\infty x^{k-1} \left(\sum_{a_1=0}^{\theta_{i_1}} \cdots \sum_{a_j=0}^{\theta_{i_j}} \frac{\Gamma(\theta_{i_1}+1)}{\Gamma(\theta_{i_1}-a_1+1)} \cdots \frac{\Gamma(\theta_{i_j}+1)}{\Gamma(\theta_{i_j}-a_j+1)} \right.$$

$$\left. \times \frac{(-1)^{\sum_{t=1}^j a_t} e^{-\sum_{t=1}^j a_t \frac{\alpha}{x}}}{\prod_{t=1}^j a_t!} dx \right),$$

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \sum_{a_1=0}^{\theta_{i_1}} \cdots \sum_{a_j=0}^{\theta_{i_j}} \left(\prod_{t=1}^j \frac{\Gamma(\theta_{i_t}+1)}{\Gamma(\theta_{i_t}-a_t+1)} \frac{(-1)^{a_t}}{a_t!} \right.$$

$$\left. \times \int_0^\infty x^{k-1} e^{-\sum_{t=1}^j a_t \frac{\alpha}{x}} dx \right)$$

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \sum_{a_1=0}^{\theta_{i_1}} \cdots \sum_{a_j=0}^{\theta_{i_j}} \left(\prod_{t=1}^j \frac{(-1)^{a_t}}{(\theta_{i_t}+1)\,\beta(a_t+1, \theta_{i_t}-a_t+1)} \right.$$

$$\left. \times \int_0^\infty x^{k-1} e^{-\sum_{t=1}^j a_t \frac{\alpha}{x}} dx \right)$$

Substituting:

$$y = \sum_{t=1}^{j} a_t \frac{\alpha}{x}$$

$$\therefore \int_{0}^{\infty} x^{k-1} e^{-\sum_{t=1}^{j} a_t \frac{\alpha}{x}} dx = \alpha^k \left(\sum_{t=1}^{j} a_t \right)^k \int_{0}^{\infty} y^{-k-1} e^{-y} dy$$

$$= \alpha^k \left(\sum_{t=1}^{j} a_t \right)^k \left(\sum_{\substack{i=0 \\ i \neq k}}^{\infty} \frac{(-1)^i}{i!(i-k)} + E_{k+1}(1) \right)$$

From **[4]**, $E_n(z)$ called the exponential integral function, which is defined as:

$$E_n(z) = \int_{1}^{\infty} \frac{e^{-zy}}{y^n} dy$$

By some arranging, we get (9).

Table 1: $I_j(k)$ using Eq. (9) when n = 3

j	$I_j(k)$
1	$I_1(k) = k\,\alpha^k \left(\sum_{\substack{i=0 \\ i \neq k}}^{\infty} \frac{(-1)^i}{i!(i-k)} + E_{k+1}(1) \right) \left(\sum_{l=1}^{3} \sum_{a_1=0}^{\theta_l} \frac{(-1)^{a_1}(a_1)^k}{(\theta_l+1)\,\beta(a_1+1,\theta_l-a_1+1)} \right)$
2	$I_2(k) = k\,\alpha^k \left(\sum_{\substack{i=0 \\ i \neq k}}^{\infty} \frac{(-1)^i}{i!(i-k)} + E_{k+1}(1) \right) \left(\sum_{a_1=0}^{\theta_1} \sum_{a_2=0}^{\theta_2} \left(\frac{(-1)^{a_1+a_2}(a_1+a_2)^k}{(\theta_1+1)(\theta_2+1)\,\beta(a_1+1,\theta_1-a_1+1)\beta(a_2+1,\theta_2-a_2+1)} \right. \right.$ $+ \sum_{a_1=0}^{\theta_1} \sum_{a_2=0}^{\theta_3} \frac{(-1)^{a_1+a_2}(a_1+a_2)^k}{(\theta_1+1)(\theta_3+1)\,\beta(a_1+1,\theta_1-a_1+1)\beta(a_2+1,\theta_3-a_2+1)}$ $\left. \left. + \sum_{a_1=0}^{\theta_2} \sum_{a_2=0}^{\theta_3} \frac{(-1)^{a_1+a_2}(a_1+a_2)^k}{(\theta_2+1)(\theta_3+1)\,\beta(a_1+1,\theta_2-a_1+1)\beta(a_2+1,\theta_3-a_2+1)} \right) \right)$

3	$I_3(k) = k\ \alpha^k \left(\sum\limits_{\substack{i=0 \\ i \neq k}}^{\infty} \dfrac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \right) \left(\sum\limits_{a_1=0}^{\theta_1} \sum\limits_{a_2=0}^{\theta_2} \sum\limits_{a_3=0}^{\theta_3} \dfrac{(-1)^{\sum\limits_{i=1}^{3} a_i}}{\prod\limits_{i=1}^{3}(\theta_i+1)\,\beta(a_i+1,\theta_i-a_i+1)}\,(\sum\limits_{i=1}^{3} a_i)^k \right)$

Corollary (3.2):

For the case of a sample from Independent Identically Distributed (IID) random variables having cdf in Eq. (7) the $I_j(k)$ in theorem (3.1) simply reduces to

$$I_j(k) = k\ \alpha^k \left(\sum_{i=0}^{\infty} \frac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \right) \binom{n}{j} \sum_{a_1=0}^{\theta} \cdots \sum_{a_j=0}^{\theta} \psi_j \left(\sum_{t=1}^{j} a_t \right)^k \qquad (13)$$

Where:

$$\psi_{i_j} = \prod_{t=1}^{j} \frac{\Gamma(\theta_t+1)}{\Gamma(\theta_t-a_t+1)} \frac{(-1)^{a_t}}{a_t!}$$

Or

$$\psi_j = \prod_{t=1}^{j} \frac{(-1)^{a_t}}{(\theta_t+1)\beta(a_t+1,\theta_t-a_t+1)}$$

Table 2: $I_j(k)$ of IID case using Eq. (13) when n = 3

j	$I_j(k)$
1	$I_1(k) = k\ \alpha^k \left(\sum\limits_{\substack{i=0 \\ i \neq k}}^{\infty} \dfrac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \right) n \sum\limits_{a_i=0}^{\theta_i} \dfrac{(-1)^{a_i}\,(a_i)^k}{(\theta_i+1)\,\beta(a_i+1,\theta_i-a_i+1)}$
2	$I_2(k) = k\ \alpha^k \left(\sum\limits_{\substack{i=0 \\ i \neq k}}^{\infty} \dfrac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \right) \binom{n}{2} \sum\limits_{a_1=0}^{\theta} \sum\limits_{a_2=0}^{\theta} \left(\dfrac{(-1)^{a_1+a_2}\,(a_1+a_2)^k}{(\theta+1)(\theta+1)\,\beta(a_1+1,\theta-a_1+1)\beta(a_2+1,\theta-a_2+1)} \right)$
3	$I_3(k) = k\ \alpha^k \left(\sum\limits_{\substack{i=0 \\ i \neq k}}^{\infty} \dfrac{(-1)^i}{i\,!(i-k)} + E_{k+1}(1) \right)$ $\binom{n}{3} \times \left(\sum\limits_{a_1=0}^{\theta} \sum\limits_{a_2=0}^{\theta} \sum\limits_{a_3=0}^{\theta} \dfrac{(-1)^{\sum\limits_{i=1}^{3} a_i}}{\prod\limits_{i=1}^{3}(\theta+1)\,\beta(a_i+1,\theta-a_i+1)}\,(\sum\limits_{i=1}^{3} a_i)^k \right)$

Numerical applications:

The following Table is computed when k = 1, θ_1 = 0.5, θ_2 =1.5, θ_3 = 2, α = { 1, 1.5, 2.5 }

Table (3): $I_j(k)$ of INID using Equations in table (1)

r	j / k	1	2	3
		1	**2**	**3**
	θ_1 = 0.5 , θ_2 =1.5 , θ_3 = 2 , α = 1			
1	1	0.10570	0.607752	0.502056
	2	-3.23112	3.05802	6.28912
	3	-16.0122	34.8447	50.8568
	θ_1 = 0.5 , θ_2 =1.5 , θ_3 = 2 , α = 1.5			
2	1	0.158544	0.911629	0.753085
	2	-7.27001	6.88054	14.1506
	3	-54.0411	117.601	171.642
	θ_1 = 0.5 , θ_2 =1.5 , θ_3 = 2 , α = 2.5			
3	1	0.26424	1.51938	1.25514
	2	-20.1945	19.1126	39.3071
	3	-250.19	544.448	794.638

For example, we can compute $\mu_{1:3}^{(1)}$ & $\mu_{2:3}^{(2)}$ using Eq. (10), Table (1) and Table (2) as:

$$\mu_{1:3}^{(1)} = I_3(1) = 0.26424.$$

$$\mu_{2:3}^{(2)} = I_2(2) - 2I_3(2) = 6.88054 - 2(14.1506) = -21.42066$$

Example (2): Set n = 2, α = 2, and θ_1 =1 (0.5) 3, θ_2 =1 (0.5) 3 in Theorem (3.1) we get:

Table 4: The values of $I_1(1)$ arising from INID GIE using Eq. (9) or Table (1)

θ_1 / θ_2	1	1.5	2	2.5	3
1	1.69114	2.11392	0.84557	-0.211392	0.84557
1.5	2.11392	2.53671	1.26835	0.211392	1.26835
2	0.845569	1.26835	0.00000	-1.05696	0.00000
2.5	-0.211392	0.211392	-1.05696	-2.11392	-1.05696
3	0.845569	1.26835	0.00000	-1.05696	0.00000

Conclusion

The order statistics of independent and non-identical distributed random variables is one of the topics that renewed by the emergence of new distributions, especially in ways of finding the moments of these distributions. Therefore, I recommend researchers to continue finding these moments especially for families of distributions by using different methods.

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