Moments of Order Statistics From Independent Nonidentical Random Variables for Group Distributions

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Abstract: In this paper we derive a general relation for the moments of order statistics (o.s.) from independent and Nonidentically distributed (inid) random variables (r.v.'s) arising from a group of distributions. This group of distributions is represented by the cumulative distribution function (c.d.f.) in the form
$F(x) = 1 - b\, e^{-m\,\lambda(x)}$, $\beta \le x \le \delta$. Application for eight known distributions is given.
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1. Introduction

Let $X_1, X_2, \dots, X_n$ be inid r.v.'s with c.d.f.'s
$F_i(x) = 1 - b\, e^{-m_i\,\lambda(x)}$, $\beta \le x \le \delta,\quad i = 1,2,3,\dots,n$. Let $X_1 \le X_2 \le \dots \le X_n$ denote the corresponding o.s. There are three known methods in the literature for deriving the moments of o.s. from inid r.v.'s.These three methods were adopted by Balakrishnan (1994a), Barakat and Abdelkader (2003), and Jamjoom and Al-Saiary (2011). The last method depends mainly on the results of the second method.

To derive the moments of o.s. from inid r.v.'s arising from this group of distributions we will also need the following theorem which was established by (Barakat and Abdelkader, 2003). They applied it to several continuous distributions such as: Erlang, Positive Exponential, Pareto, and Laplace distribution. Else it used to compute the moments of INID o.s from Erlang distribution (Abdelkader, 2003), Gamma distribution (Abdelkader, 2004), Burr type XII (Jamjoom, 2006) and Beta distribution (Abdelkader, 2008).

Theorem 1: Let $X_1, X_2, \cdots, X_n$ be independent nonidentically distributed r.v.s. The k^{th} moment of all order statistics, $\mu_{r:n}^{(k)}$,for $1 \le r \le n$ and $k = 1, 2, \dots$ is given by:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \qquad (1)$$

Where:

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} G_{i_t}(x)\, dx, \quad j = 1, 2, \cdots, n \qquad (2)$$

$G_{i_t}(x) = 1 - F_{i_t}(x)$, with $\left(i_1, i_2, \cdots, i_n\right)$ is a permutation of (1, 2, …, n) for which $i_1 \le i_2 < \cdots < i_n$.

Proof: The proof of this theorem can be found in Barakat and Abdelkader (2003).

Theorem 2: Let $X_1, X_2, \cdots, X_n$ be independent nonidentically distributed r.v.s. drawn from group of distributions with the formula

$$F_i(x) = 1 - b\, e^{-m_i\, \lambda(x)}, \ \beta \le x \le \delta$$

The k^{th} moment of all order statistics, $\mu_{r:n}^{(k)}$,for $1 \le r \le n$ and $k = 1, 2, \dots$ is given by:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \left[I_j(k+1) - \binom{n}{j} \beta^k \right] \quad (3)$$

Where

$$I_j'(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum b^j \sum_{t=1.}^{j} m_{i_t}$$

$$\times \int_{\beta}^{\delta} x^k \lambda'(x) e^{-\lambda(x) \sum_{t=1}^{j} m_{i_t}} dx, \quad j = 1, 2, \cdots, n \quad (4)$$

Where

$0 \le \beta \le x \le \delta \le \infty$

Proof

From theorem 1:

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} I_j(k) \quad (5)$$

where

$$I_j(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum k \int_0^{\infty} x^{k-1} \prod_{t=1}^{j} \left[1 - F_{i_t}(x) \right] dx, \quad j = 1, 2, \cdots, n \quad (6)$$

Let

$$F_i(x) = 1 - b\, e^{-m_i \lambda(x)}, \quad \beta \le x \le \delta \quad (7)$$

$$F(\beta) = 0 \Rightarrow 0 = 1 - b\, e^{-m_i \lambda(\beta)}$$

$$\Rightarrow b\, e^{-m_i \lambda(\beta)} = 1$$

$$\Rightarrow e^{-m_i \lambda(\beta)} = \frac{1}{b}$$

$$\Rightarrow \prod_{t=1}^{j} e^{-m_{i_t} \lambda(\beta)} = \prod_{t=1}^{j} \frac{1}{b}$$

$$\Rightarrow e^{-\lambda(\beta) \sum_{t=1}^{j} m_{i_t}} = \frac{1}{b^j} = b^{-j} \quad (8)$$

$$F(\delta)=1 \Rightarrow 1 = 1-b\,e^{-m_i \lambda(\delta)}$$
$$\Rightarrow b\,e^{-m_i \lambda(\delta)} = 0$$
$$\Rightarrow e^{-m_i \lambda(\delta)} = 0$$
$$\Rightarrow \prod_{t=1}^{j} e^{-m_{i_t} \lambda(\delta)} = 0$$
$$\Rightarrow e^{-\lambda(\delta)\sum_{t=1}^{j} m_{i_t}} = 0 \qquad (9)$$

Substituting from (9) in (6) we obtain:

$$I_j(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum k \int_{\beta}^{\delta} x^{k-1} \prod_{t=1}^{j} b\,e^{-m_{i_t} \lambda(x)}\,dx$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum k\, b^j \int_{\beta}^{\delta} x^{k-1} \prod_{t=1}^{j} e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx$$

Let:

$$u = e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}$$

$$du = -\lambda'(x)\sum_{t=1}^{j} m_{i_t}\, e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx$$

$$dv = x^{k-1}\,dx$$

$$v = \frac{x^k}{k}$$

$$I_j(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum k\, b^j \Bigg[\frac{x^k}{k} e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}} \Bigg|_{\beta}^{\delta}$$

$$+ + \frac{\sum_{t=1}^{j} m_{i_t}}{k} \int_{\beta}^{\delta} x^k\, \lambda'(x)\, e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx \Bigg]$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum b^j \left[\delta^k e^{-\lambda(\delta)\sum_{t=1}^{j} m_{i_t}} - \beta^k e^{-\lambda(\beta)\sum_{t=1}^{j} m_{i_t}} \right]$$

$$+ \sum_{1\le i_1<i_2<\cdots<i_j\le n} \cdots \sum b^j \left[\sum_{t=1}^{j} m_{i_t} \int_{\beta}^{\delta} x^k\, \lambda'(x)\, e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx \right]$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(-\beta^k)+\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum b^j\,[\sum_{t=1}^{j} m_{i_t}$$

$$\times\int_{\beta}^{\delta} x^k\,\lambda'(x)e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx\,] \qquad (10)$$

$$=-\binom{n}{j}\beta^k+I_j'(k)$$

Where

$$I_j'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum b^j\sum_{t=1}^{j} m_{i_t}\int_{\beta}^{\delta} x^k\,\lambda'(x)e^{-\lambda(x)\sum_{t=1}^{j} m_{i_t}}\,dx \qquad (11)$$

Substituting from (10) and (11) in (5) the proof finished.

For Exponential distribution

$$F(x)=1-e^{-\frac{x}{\theta}},\qquad 0\le x\le\infty$$

In (9) put $b=1,\ m=\frac{1}{\theta},\ \lambda(x)=x,\lambda'(x)=1,\beta=0,\ \delta=\infty$ we get

$$I_j'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(1)^j\,(\sum_{t=1}^{j}\frac{1}{\theta i_t})\int_0^\infty x^k\,(1)e^{-x\sum_{t=1}^{j}\frac{1}{\theta i_t}}\,dx$$

$$=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(\sum_{t=1}^{j}\frac{1}{\theta i_t})\int_0^\infty x^k\,e^{-x\sum_{t=1}^{j}\frac{1}{\theta i_t}}\,dx$$

$$=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{\Gamma(k+1)}{(\sum_{t=1}^{j}\frac{1}{\theta i_t})^k}$$

$$\therefore\ \mu_{r:n}^{(k)}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$

$$\times\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{k!}{(\sum_{t=1}^{j}\frac{1}{\theta i_t})^k} \qquad (12)$$

This result was obtained by Barakat & Abdelkader (2003).

For Beta type 1

$$F(x)=1-[\frac{\delta-x}{\delta-\beta}]^{p_i},\qquad \beta\le x\le\delta,\ p_i>0,\ \delta,\ \beta>0 \qquad (13)$$

We can put (11) as

$$F(x)=1-e^{p_i \ln[\frac{\delta-x}{\delta-\beta}]},\ \beta\le x\le\delta, p_i>0,\ \delta,\beta>0 \tag{14}$$

In (9) put $b=1,\ m_i=-p_i,\ \lambda(x)=\ln[\frac{\delta-x}{\delta-\beta}], \lambda'(x)=-\frac{1}{\delta-x}$, we get

$$I_j{}'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(-\sum_{t=1}^{j}p_{i_t})\int_{\beta}^{\delta}x^k(-\frac{1}{\delta-x})e^{\sum_{t=1}^{j}p_{i_t}\ln[\frac{\delta-x}{\delta-\beta}]}dx$$

$$I_j{}'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(\sum_{t=1}^{j}p_{i_t})\int_{\beta}^{\delta}x^k[\frac{\delta-x}{\delta-\beta}]^{\sum_{t=1}^{j}p_{i_t}}\frac{1}{\delta-x}dx$$

$$=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{(\sum_{t=1}^{j}p_{i_t})}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}\int_{\beta}^{\delta}x^k[\delta-x]^{\sum_{t=1}^{j}p_{i_t}-1}dx$$

$$=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{(\sum_{t=1}^{j}p_{i_t})\delta^{\sum_{t=1}^{j}p_{i_t}-1}}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}\int_{\beta}^{\delta}x^k[1-\frac{x}{\delta}]^{\sum_{t=1}^{j}p_{i_t}-1}dx$$

Substituting $y=1-\frac{x}{\delta}$

$$I_j{}'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{(\sum_{t=1}^{j}p_{i_t})\delta^{\sum_{t=1}^{j}p_{i_t}-1+k}}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}$$

$$\times\int_{\frac{\delta-\beta}{\delta}}^{0}(1-y)^k\,y^{\sum_{t=1}^{j}p_{i_t}-1}(-\delta dy)$$

$$I_j'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{\left(\sum_{t=1}^{j}p_{i_t}\right)\delta^{\sum_{t=1}^{j}p_{i_t}+k}}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}\int_0^{\frac{\delta-\beta}{\delta}}y^{\sum_{t=1}^{j}p_{i_t}-1}(1-y)^k\,dy$$

$$=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{\left(\sum_{t=1}^{j}p_{i_t}\right)\delta^{\sum_{t=1}^{j}p_{i_t}+k}}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}\beta_{\frac{\delta-\beta}{\delta}}\left(\sum_{t=1}^{j}p_{i_t},k+1\right)$$

$$\therefore\ \mu_{r:n}^{(k)}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$

$$\times\left[\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{\left(\sum_{t=1}^{j}p_{i_t}\right)\delta^{\sum_{t=1}^{j}p_{i_t}+k}}{(\delta-\beta)^{\sum_{t=1}^{j}p_{i_t}}}\right.$$

$$\left.\times\beta_{\frac{\delta-\beta}{\delta}}\left(\sum_{t=1}^{j}p_{i_t},k+1\right)-\binom{n}{j}\beta^k\right]\qquad(15)$$

This result was obtained by Jamjoom & Al-saiary (2010).

For Weibull distribution

Let $b=1,\ m_i=\frac{1}{\theta_i},\ \lambda(x)=x^p,\ \lambda'(x)=p\,x^{p-1},\ \beta=0,\ \delta=\infty$

$$I_j'(k)=-\binom{n}{j}(0)+\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(1)^j$$

$$\times\left(\sum_{t=1}^{j}\frac{1}{\theta i_t}\right)p\int_0^{\infty}x^k\,x^{p-1}e^{-x^p\sum_{t=1}^{j}\frac{1}{\theta i_t}}\,dx$$

Substituting $y=x^p\sum_{t=1}^{j}\frac{1}{\theta i_t}$ **we get**

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{1}{\left(\sum_{t=1}^{j}\frac{1}{\theta i_t}\right)^{\frac{k}{p}}}\int_0^\infty y^{\frac{k}{p}} e^{-y}\,dy$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma\left(\frac{k}{p}+1\right)}{\left(\sum_{t=1}^{j}\frac{1}{\theta i_t}\right)^{\frac{k}{p}}}$$

$$\mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$

$$\times \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\frac{k}{P}!}{\left(\sum_{t=1}^{j}\frac{1}{\theta i_t}\right)^{\frac{k}{p}}} \qquad (16)$$

This result was obtained Barakat & Abdelkader (2000).

For Rayleigh distribution

By put p = 2 in (15) we get

$$\therefore\ \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$

$$\times \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\frac{k}{2}!}{\left(\sum_{t=1}^{j}\frac{1}{\theta i_t}\right)^{\frac{k}{2}}} \qquad (17)$$

This result was obtained by Barakat & Abdelkader (2003).

For Pareto Distribution

$$F(x) = 1 - x^{-\nu} \qquad 1 \le x \le \infty$$

$$= 1 - e^{-\nu \ln x}$$

Let $b=1,\ m_i = \nu_i,\ \lambda(x)=\ln(x),\ \lambda'(x)=\frac{1}{x},\ \ \beta=1,\ \delta=\infty$

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \left(\sum_{t=1}^{j}\nu_i\right)\int_1^\infty x^{k-1} e^{-\ln x \sum_{t=1}^{j}\nu_{i_{i_t}}}\,dx$$

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \left(\sum_{t=1}^{j}\nu_i\right)\int_\beta^\delta x^{k-\sum_{t=1}^{j}\nu_i - 1}\,dx$$

Let $\sum_{t=1}^{j} \nu_i > k$

$$I_j'(k) = \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \left(\sum_{t=1}^{j} \nu_i\right) \left[\frac{x^{k - \sum_{t=1}^{j} \nu_i - 1}}{k - \sum_{t=1}^{j} \nu_i} \right]_1^{\infty}$$

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \left(\sum_{t=1}^{j} \nu_i\right) \left[1 - \frac{1}{k - \sum_{t=1}^{j} \nu_i} \right]$$

$$= \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{\sum_{t=1}^{j} \nu_i}{\sum_{t=1}^{j} \nu_i - k}, \qquad \sum_{t=1}^{j} \nu_i > k$$

$$\therefore \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \left[\sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{\sum_{t=1}^{j} \nu_i}{\sum_{t=1}^{j} \nu_i - k} - \binom{n}{j} \right]$$

This result was obtained by Barakat & Abdelkader (2003).

For Exponentiated Frechet distribution:

$$F_i(x) = 1 - \left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}} \right]^{\alpha_i}, \qquad x > 0,\ \sigma > 0,\ \lambda > 0,\ \alpha > 0 \tag{18}$$

See Nadarajah, S. & Kotz, S. (2003) and Badr, M. (2010).

We can put Eq (18) as:

$$F(x) = 1 - e^{\alpha_i \ln\left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right]}, \qquad x > 0, \sigma > 0,\ \lambda > 0,\ \alpha_i > 0 \tag{19}$$

$$Let \quad b = 1,\ m_i = -\alpha_i,\ \lambda(x) = \ln\left[1 - e^{-\left(\frac{\sigma}{x}\right)^{\lambda}}\right],\ \lambda'(x) = -\frac{-\lambda \sigma^{\lambda} x^{-\lambda - 1}}{e^{\left(\frac{\sigma}{x}\right)^{\lambda}} - 1} \ in\ (9)\ we\ \ get$$

$$I_j'(k)=\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\left(-\sum_{t=1}^{j}\alpha_{i_t}\right)\int_0^\infty x^k\left(-\frac{\lambda\sigma^\lambda x^{-\lambda-1}}{e^{(\frac{\sigma}{x})^\lambda}-1}\right)$$

$$\times e^{\sum_{t=1}^{j}\alpha_{i_t}\ln\left[1-e^{-(\frac{\sigma}{x})^\lambda}\right]}dx$$

$$I_j'(k)=\lambda\sigma^\lambda\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\left(\sum_{t=1}^{j}\alpha_{i_t}\right)\int_0^\infty x^{k-\lambda-1}\left[1-e^{-(\frac{\sigma}{x})^\lambda}\right]^{\sum_{t=1}^{j}\alpha_{i_t}-1}$$

$$\times e^{-(\frac{\sigma}{x})^\lambda}dx$$

By using binomial expanding we obtain:

$$I_j'(k)=\lambda\sigma^\lambda\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\left(\sum_{t=1}^{j}\alpha_{i_t}\right)\int_0^\infty x^{k-\lambda-1}$$

$$\times\left(\sum_{l=0}^{\sum_{t=1}^{j}\alpha_{i_t}-1}\binom{\sum_{t=1}^{j}\alpha_{i_t}-1}{l}(-1)^l e^{-(l+1)(\frac{\sigma}{x})^\lambda}\right)dx$$

$$\therefore\ \mu_{r:n}^{(k)}=\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r}$$

$$\times\left[\lambda\sigma^\lambda\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\left(\sum_{t=1}^{j}\alpha_{i_t}\right)\right.$$

$$\left.\times\left(\sum_{l=0}^{\sum_{t=1}^{j}\alpha_{i_t}-1}\binom{\sum_{t=1}^{j}\alpha_{i_t}-1}{l}(-1)^l\int_0^\infty x^{k-\lambda-1}e^{-(l+1)(\frac{\sigma}{x})^\lambda}dx\right)\right]$$

Let

$$y=(l+1)\left(\frac{\sigma}{x}\right)^\lambda\Rightarrow x=\sigma(l+1)^{\frac{1}{\lambda}}y^{\frac{1}{\lambda}}$$

$$dx=-\frac{\sigma}{\lambda}(l+1)^{\frac{1}{\lambda}}y^{-\frac{1}{\lambda}-1}dy$$

$$\therefore\ \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \Big[\sigma^{\lambda} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \Big(\sum_{t=1}^{j} \alpha_{i_t} \Big)$$

$$\times \left(\sum_{l=0}^{\sum_{t=1}^{j} \alpha_{i_t} - 1} \binom{\sum_{t=1}^{j} \alpha_{i_t} - 1}{l} \frac{(-1)^l}{(l+1)^{\frac{k}{\lambda}-1}} \int_0^{\infty} y^{-\frac{k}{\lambda}+1-1} e^{-y} dy \right) \Big]$$

$$\therefore\ \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \Big[\sigma^{\lambda} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \Big(\sum_{t=1}^{j} \alpha_{i_t} \Big)$$

$$\times \left(\sum_{l=0}^{\sum_{t=1}^{j} \alpha_{i_t} - 1} \binom{\sum_{t=1}^{j} \alpha_{i_t} - 1}{l} \frac{(-1)^l}{(l+1)^{\frac{k}{\lambda}-1}} \int_0^{\infty} y^{-\frac{k}{\lambda}+1-1} e^{-y} dy \right) \Big]$$

$$\therefore\ \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \Big[\sigma^{\lambda} \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \Big(\sum_{t=1}^{j} \alpha_{i_t} \Big)$$

$$\times \left(\sum_{l=0}^{\sum_{t=1}^{j} \alpha_{i_t} - 1} \binom{\sum_{t=1}^{j} \alpha_{i_t} - 1}{l} (-1)^l \frac{\Gamma(1-\frac{k}{\lambda})}{(l+1)^{\frac{k}{\lambda}-1}} \right) \Big] \qquad (20)$$

This result was obtained by Jamjoom & Al-saiary (2011 to appear).

For Erlang truncated Exponential distribution:

$$F(x) = 1 - e^{-\alpha x (1-e^{-\lambda})}, \qquad 0 \le x \le \infty, \alpha, \lambda > 0$$

$b=1,\ m=\beta(1-e^{-\lambda}),\ \lambda(x)=x, \lambda'(x)=1, \beta=0,\ \delta=\infty$ Let

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum (1)^j \left(\sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})\right)$$

$$\times \int_0^\infty x^k (1) e^{-x \sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})} dx$$

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \left(\sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})\right)$$

$$\times \int_0^\infty x^k e^{-x \sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})} dx$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma(k+1)}{\left(\sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})\right)^k}$$

$$\therefore \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$

$$\times \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma(k+1)}{\left(\sum_{t=1}^{j} \alpha (1-e^{-\lambda_{i_t}})\right)^k} \qquad (21)$$

This result was obtained by Jamjoom & Al-saiary (2010) and (2011).

For Laplace Distribution

$$F_i(x) = 1-\frac{1}{2}e^{-\alpha_i x} \qquad x>0.\ \alpha>0 \qquad (22)$$

In (9) put $b=\frac{1}{2},\ m=\alpha,\ \lambda(x)=x,\lambda'(x)=1, \beta=0,\ \delta=\infty$ we get

$$I_j'(k) = \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \left(\frac{1}{2}\right)^j \left(\sum_{t=1}^{j} \alpha_{i_t}\right) \int_0^\infty x^k e^{-x \sum_{t=1}^{j} \alpha_{i_t}} dx$$

$$= \sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum \frac{\Gamma(k+1)}{2^j \left(\sum_{t=1}^{j} \alpha_{i_t}\right)^k}$$

$$\therefore \mu_{r:n}^{(k)} = \sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r}$$
$$\times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{k!}{2^j \, (\sum_{t=1}^{j} \alpha_{i_t})^k} \qquad (23)$$

This result was obtained by Barakat & Abdelkader (2003).

Table 1. The k^th moments of the r^th o.s. from independent nonidentical random variables for some distributions

Distribution	F(x)	$\lambda(x)$	$\lambda'(x)$	β	δ	b	m_i	$\mu_{r:n}^{(k)}$
Exponential	$1-e^{-\frac{x}{\theta}}$	x	1	0	∞	1	$\frac{1}{\theta_i}$	$\sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{k!}{(\sum \frac{1}{\theta_{i_t}})^k}$
Weibull	$1-e^{-\frac{x^p}{\theta}}$	x^p	$p x^{p-1}$	0	∞	1	$\frac{1}{\theta_i}$	$\sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{\frac{k}{P}!}{(\sum_{t=1}^{j} \frac{1}{\theta i_t})^{\frac{k}{p}}}$
Beta type I	$1-[\frac{\delta-x}{\delta-\beta}]^{p_i}$	$\ln[\frac{\delta-x}{\delta-\beta}]$	$\frac{-1}{\delta-x}$	β	δ	1	$-p_i$	$\sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \times [\sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum [\frac{(\sum_{t=1}^{j} p_{i_t}) \delta^{\sum_{t=1}^{j} p_{i_t} + k}}{(\delta-\beta)^{\sum_{t=1}^{j} p_{i_t}}} \times \beta_{\frac{\delta-\beta}{\delta}}(\sum_{t=1}^{j} p_{i_t}, k+1) - \binom{n}{j} \beta^k]$
Rayleigh	$1-e^{-\frac{x^2}{\theta}}$	x^2	$2x$	0	∞	1	$\frac{1}{\theta_i}$	$\sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{\frac{k}{2}!}{(\sum_{t=1}^{j} \frac{1}{\theta i_t})^{\frac{k}{2}}}$
Pareto	$F(x)=1-x^{-\nu}$	$\ln(x)$	$\frac{1}{x}$	1	∞	1	ν_i	$\sum_{j=n-r+1}^{n} (-1)^{j-(n-r+1)} \binom{j-1}{n-r} \times \sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum [\sum_{1 \le i_1 < i_2 < \cdots < i_j \le n} \cdots \sum \frac{\sum_{t=1}^{j} \nu_i}{\sum_{t=1}^{j} \nu_i - k} - \binom{n}{j}]$

Distribution	F(x)	$\lambda(x)$	$\lambda'(x)$	β	δ	b	m_i	$\mu_{r:n}^{(k)}$
Exponentiated Frechet	$1-\left[1-e^{-(\frac{\sigma}{x})^{\lambda}}\right]^{\alpha}$	$\ln\left[1-e^{-(\frac{\sigma}{x})^{\lambda}}\right]$	$\frac{-\lambda\sigma^{\lambda}x^{-\lambda-1}}{e^{(\frac{\sigma}{x})^{\lambda}}-1}$	0	∞	1	$-\alpha_i$,	$\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r} \times[\sigma^{\lambda}\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum(\sum_{t=1}^{j}\alpha_{i_t}) \times\left(\sum_{l=0}^{\sum_{t=1}^{j}\alpha_{i_t}-1}\binom{\sum_{t=1}^{j}\alpha_{i_t}-1}{l}(-1)^{l}\frac{\Gamma(1-\frac{k}{\lambda})}{(l+1)^{\frac{k}{\lambda}-1}}\right)]$
Erlang Truncated Exponential	$1-e^{-\beta x(1-e^{-\lambda})}$	x	1	0	∞	1	$\alpha\times(1-e^{-\lambda_i})$	$\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r} \times\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{\Gamma(k+1)}{(\sum_{t=1}^{j}\alpha(1-e^{-\lambda_{i_t}}))^{k}}$
Laplace Distribution	$1-\frac{1}{2}e^{-\alpha x}$	x	1	0	∞	$\frac{1}{2}$	α_i	$\sum_{j=n-r+1}^{n}(-1)^{j-(n-r+1)}\binom{j-1}{n-r} \times\sum_{1\le i_1<i_2<\cdots<i_j\le n}\cdots\sum\frac{k!}{2^{j}(\sum_{t=1}^{j}\alpha_{i_t})^{k}}$

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